Matematika G2 - GY14 Sándor Tibor

1. feladat

a) $\iint_T \frac{1}{\sqrt{x^2+y^2+1}}\, dT \qquad T=\{(x;y) \mid x^2+y^2\leq 1,\ x;y\geq 0\}$

Hagyományosan: $x\in[0;1] \quad y\in[0;\sqrt{1-x^2}]$

Polárkoordináta: $x=r\cos\varphi \quad y=r\sin\varphi$

$$\underline{\underline{J}}=\begin{bmatrix}\cos\varphi & -r\sin\varphi\\ \sin\varphi & r\cos\varphi\end{bmatrix} \Rightarrow \det\underline{\underline{J}}=r$$

$r\in[0;1] \quad \varphi\in[0;\pi/2]$

$$\int_0^1\int_0^{\pi/2}\frac{1}{\sqrt{r^2+1}}\, r\, d\varphi\, dr=\frac{\pi}{4}\int_0^1 \underbrace{2r}_{g'}\cdot\underbrace{(r^2+1)^{-1/2}}_{f\circ g}\, dr=$$

$$=\frac{\pi}{4}\cdot 2\,(r^2+1)^{1/2}\Big|_0^1=\frac{\pi}{2}(\sqrt{2}-1)$$

b) $f(x;y)=x^2+y^2 \qquad T=\{(x;y)\mid (x-3)^2+(y-2)^2\leq 1\}$

$$\begin{cases}x=3+r\cos\varphi\\ y=2+r\sin\varphi\end{cases} \qquad \det\underline{\underline{J}}=r \qquad r\in[0;1] \quad \varphi\in[0;2\pi]$$

$$f(r;\varphi)=(3+r\cos\varphi)^2+(2+r\sin\varphi)^2=13+6r\cos\varphi+4r\sin\varphi+r^2$$

$$\int_0^1\int_0^{2\pi} 13r+\cancel{6r\cos\varphi}+\cancel{4r\sin\varphi}+r^3\, d\varphi\, dr=\int_0^1 26\pi r+2\pi r^3\, dr=$$

$$=13\pi+\frac{\pi}{2}=\frac{27\pi}{2}$$

c) $f(x;y) = |2xy| \quad T = \{ \frac{x^2}{9} + \frac{y^2}{4} \leq 1 \wedge x;y \geq 0 \}$

$$\begin{cases} x = 3r\cos\varphi & r \in [0;1] \\ y = 2r\sin\varphi & \varphi \in [0;\pi/2] \end{cases}$$

$$\underline{\underline{J}} = \begin{bmatrix} 3\cos\varphi & -3r\sin\varphi \\ 2\sin\varphi & 2r\cos\varphi \end{bmatrix} \Rightarrow \det \underline{\underline{J}} = 6r$$

$$\int_0^{\pi/2} \int_0^1 2 \cdot 3r\cos\varphi \cdot 2r\sin\varphi \cdot 6r \, dr \, d\varphi =$$

$$\int_0^{\pi/2} \int_0^1 72r^3 \cos\varphi \sin\varphi \, dr \, d\varphi = \int_0^{\pi/2} 18 \underbrace{\cos\varphi}_{f'} \underbrace{\sin\varphi}_{f} \, d\varphi =$$

$$= 9\sin^2\varphi \Big|_0^{\pi/2} = 9$$

d) $f(x;y) = 4xy^3 \quad T = \{ 1 \leq x^2 + y^2 \leq 4 \,;\, x \geq 0 \,;\, y \geq \frac{x}{\sqrt{3}} \}$

$$\begin{cases} x = r\cos\varphi & r \in [1;2] \\ y = r\sin\varphi & \varphi \in [\pi/6;\pi/2] \end{cases} \qquad \det \underline{\underline{J}} = r$$

$$\int_{\pi/6}^{\pi/2} \int_1^2 4r\cos\varphi \, r^3 \sin^3\varphi \, r \, dr \, d\varphi = \int_{\pi/6}^{\pi/2} \int_1^2 4r^5 \cos\varphi \sin^3\varphi \, dr \, d\varphi$$

$$= \int_{\pi/6}^{\pi/2} \frac{4}{6}(2^6 - 1) \cos\varphi \sin^3\varphi \, d\varphi = \int_{\pi/6}^{\pi/2} 42 \underbrace{\cos\varphi}_{f'} \underbrace{\sin^3\varphi}_{f^3} \, d\varphi$$

$$= 42 \frac{\sin^4\varphi}{4} \Big|_{\pi/6}^{\pi/2} = \frac{42}{4}\left(1 - \frac{1}{2^4}\right) = \frac{21}{2} \cdot \frac{15}{16} = \frac{315}{32}$$

e) $f(x;y) = \sin(x^2+y^2)$ $\quad T = \{x^2+y^2 \leq 2^2;\ x,y > 0\}$

$x = r\cos\varphi$ $\quad r \in [0;2]$ $\quad \det \underline{\underline{J}} = r$

$y = r\sin\varphi$ $\quad \varphi \in [0; \pi/2]$

$$\int_0^2 \int_0^{\pi/2} \sin(r^2) r \, d\varphi \, dr = \int_0^2 \frac{\pi}{2} \sin(r^2) \, r \, dr = \frac{\pi}{4} \int_0^2 2r \sin r^2 \, dr$$

$$= \frac{\pi}{4} (-\cos r^2) \Big|_0^2 = \frac{\pi}{4} (1 - \cos 4)$$

2. feladat

a) $$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} 2^{-x^2-y^2} \, dx \, dy$$

$$\int_0^{\infty} \int_0^{2\pi} 2^{-r^2} r \, d\varphi \, dr = \int_0^{\infty} (-\pi)(-2r) \, 2^{-r^2} \, dr =$$

$$-\pi \cdot 2^{-r^2} \ln 2 \Big|_0^{\infty} = \pi \cdot \ln 2$$

b) $\int_0^{\infty} e^{-x^2} dx \Rightarrow$ Gauss-görbe
elemi úton nem integrálható!

$$J^2 = \int_0^{\infty} e^{-x^2} dx \cdot \int_0^{\infty} e^{-y^2} dy = \int_0^{\infty} \int_0^{\infty} e^{-x^2-y^2} dx \, dy =$$

$$\int_0^{\infty} \int_0^{\pi/2} -\frac{1}{2} (2r) \, e^{-r^2} d\varphi \, dr = \int_0^{\infty} -\frac{\pi}{4} (2r) e^{-r^2} dr =$$

$$= -\frac{\pi}{4} e^{-r^2} \Big|_0^{\infty} = \frac{\pi}{4} \Rightarrow J = \frac{\sqrt{\pi}}{2}$$

3. feladat

$$V = \int_{x^2+y^2+z^2 \leq R} dV$$

$$\begin{cases} x = r\sin\vartheta\cos\varphi \\ y = r\sin\vartheta\sin\varphi \\ z = r\cos\vartheta \end{cases} \qquad \begin{matrix} r \in [0;R] \\ \vartheta \in [0;\pi] \\ \varphi \in [0;2\pi] \end{matrix}$$

$$dV = r^2 \sin\vartheta \, dr\, d\vartheta\, d\varphi$$

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} r^2 \sin\vartheta \, dr\, d\vartheta\, d\varphi = \int_0^{2\pi}\int_0^{\pi} \frac{R^3}{3}\sin\vartheta \, d\vartheta\, d\varphi$$

$$= \int_0^{2\pi} \frac{R^3}{3}(-\cos\vartheta)\Big|_0^{\pi} d\varphi = \int_0^{2\pi} \frac{R^3}{3}(1+1)\, d\varphi = \frac{4R^3\pi}{3}$$

4. feladat

$$f(x;y;z) = z\sqrt{x^2+y^2} \qquad T = \{x^2+y^2 \leq 1;\ 0 \leq z \leq 2;\ x;y > 0\}$$

$$\begin{matrix} x = r\cos\varphi \\ y = r\sin\varphi \\ z = z \end{matrix} \qquad \begin{matrix} r \in [0;1] \\ \varphi \in [0;\pi/2] \\ z \in [0;2] \end{matrix} \qquad \det \underline{\underline{J}} = r$$

$$\int_0^2\int_0^1\int_0^{\pi/2} z r^2 \, d\varphi\, dz\, dr = \frac{\pi}{2} \cdot \frac{1}{3} \cdot \frac{4}{2} = \frac{\pi}{3}$$

5. feladat

$$f(x;y;z)=\sqrt{x^2+y^2+z^2} \qquad T=\{x^2+y^2+z^2\le 1\}$$

$$x = r\sin\vartheta\cos\varphi \qquad r\in[0;1] \qquad \det \underline{\underline{J}} = r^2\sin\vartheta$$
$$y = r\sin\vartheta\sin\varphi \qquad \vartheta\in[0;\pi]$$
$$z = r\cos\vartheta \qquad \varphi\in[0;2\pi]$$

$$x^2+y^2+z^2=r^2$$

$$\int_0^{\pi}\int_0^{2\pi}\int_0^{1} r^3\sin\vartheta\, dr\, d\varphi\, d\vartheta = \int_0^{\pi}\frac{1}{4}\cdot 2\pi\cdot\sin\vartheta\, d\vartheta =$$

$$=\frac{\pi}{2}(-\cos\vartheta)\Big|_0^{\pi} = \pi$$

6. feladat

$$\underline{r}(t)=\begin{bmatrix} t^2+1 \\ 5t \\ 3t-2 \end{bmatrix} \qquad \underline{v}=\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \underline{s}(t;s)=\underline{r}(t)+s\underline{v}$$

$$t;s\in\mathbb{R}$$

$$\underline{s}(t;s)=\begin{bmatrix} t^2+1 \\ 5t \\ 3t-2 \end{bmatrix}+s\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

7. feladat

$$\underline{s}(t;s)=\left.\begin{bmatrix} 2\cos t \\ 2\sin t \\ s \end{bmatrix}\right\} x^2+y^2=2^2 \qquad \begin{matrix} t\in[0;2\pi] \\ s\in\mathbb{R} \end{matrix}$$

8. feladat

$$\left.\begin{matrix} x(t)=5+3\cos t \\ z(t)=3\sin t \end{matrix}\right\} \quad s(t;s)=\begin{bmatrix} (5+3\cos t)\cos s \\ (5+3\cos t)\sin s \\ 3\sin t \end{bmatrix} \qquad \begin{matrix} t\in[0;2\pi] \\ s\in[0;2\pi] \end{matrix}$$