Matematika G2 - GY12 Sándor Tibor

1. feladat

$$f(x;y) = 2x^2 - xy - y^2 - 6x - 3y + 5$$

$T_1 = ?$ $T_2 = ?$ $P(1;-2)$

$$\frac{\partial f}{\partial x} = 4x - y - 6 \Rightarrow \left.\frac{\partial f}{\partial x}\right|_P = 4 + 2 - 6 = 0$$

$$\frac{\partial f}{\partial y} = -x - 2y - 3 \Rightarrow \left.\frac{\partial f}{\partial y}\right|_P = -1 + 4 - 3 = 0$$

$$T_0 = f(P) = 2 + 2 - 4 - 6 + 6 + 5 = 5$$

$$T_1 = T_0 + \left.\frac{\partial f}{\partial x}\right|_P (x - x_0) + \left.\frac{\partial f}{\partial y}\right|_P (y - y_0) = 5 + 0(x-1) + 0(y+2) = 5$$

$$\frac{\partial^2 f}{\partial x^2} = 4 \qquad \frac{\partial^2 f}{\partial y^2} = -2 \qquad \frac{\partial^2 f}{\partial x \partial y} = -1$$

$$T_2 = T_1 = \frac{1}{2}\left[\left.\frac{\partial^2 f}{\partial x^2}\right|_P (x-x_0)^2 + \left.\frac{\partial^2 f}{\partial y^2}\right|_P (y-y_0)^2 + \left.\frac{\partial^2 f}{\partial x \partial y}\right|_P (x-x_0)(y-y_0)\right]$$

$$= 5 + \frac{1}{2}\left[4(x-1)^2 - 2(y+2)^2 - 1(x-1)(y+2)\right]$$

$$= 5 + 2(x-1)^2 - (y+2)^2 - \frac{1}{2}(x-1)(y+2)$$

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2. feladat

$$f(x;y;z) = \sin x \sin y \sin z \qquad P(\pi/4; \pi/4; \pi/6)$$

$$\frac{\partial f}{\partial x} = \cos x \sin y \sin z \Rightarrow 1/4$$

$$\frac{\partial f}{\partial y} = \sin x \cos y \sin z \Rightarrow 1/4$$

$$\frac{\partial f}{\partial z} = \sin x \sin y \cos z \Rightarrow \sqrt{3}/4$$

$$T_0 = 1/\sqrt{2} \cdot 1/\sqrt{2} \cdot 1/2 = 1/4$$

$$T_1 = \frac{1}{4} + \frac{1}{4}(x - \pi/4) + \frac{1}{4} \cdot (y - \pi/4) + \frac{\sqrt{3}}{4}(z - \pi/6)$$

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_P = -\frac{1}{4} \qquad \left.\frac{\partial^2 f}{\partial y^2}\right|_P = -\frac{1}{4} \qquad \frac{\partial^2 f}{\partial z^2} = -\frac{1}{4}$$

$$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_P = \frac{1}{4} \qquad \frac{\partial^2 f}{\partial y \partial z} = \sqrt{3}/4 \qquad \frac{\partial^2 f}{\partial z \partial x} = \frac{\sqrt{3}}{4}$$

$$T_2 = T_1 + \frac{1}{2}\Big[\frac{-1}{4}(x-x_0)^2 + \frac{-1}{4}(y-y_0)^2 + \frac{-1}{4}(z-z_0)^2 +$$
$$+ \frac{1}{4}(x-x_0)(y-y_0) + \frac{\sqrt{3}}{4}(y-y_0)(z-z_0) + \frac{\sqrt{3}}{4}(z-z_0)(x-x_0)\Big]$$

3. feladat

$$f(x;y) = x^2 - xy + y^2 + 3x - 2y + 1$$

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 2x - y + 3 \\ \frac{\partial f}{\partial y} &= -x + 2y - 2 \end{aligned}\right\}$$

Szükségesség: $\partial_i f = 0$

$2x - y + 3 = 0 \quad (\text{I})$

$-x + 2y - 2 = 0 \quad (\text{II})$

$(\text{I}) + 2(\text{II}) \Rightarrow 3y - 1 = 0 \Rightarrow y = \frac{1}{3} \quad x = -\frac{4}{3}$

Hesse:

$$\frac{\partial^2 f}{\partial x^2} = 2 \quad \frac{\partial^2 f}{\partial y^2} = 2 \quad \frac{\partial^2 f}{\partial x \partial y} = -1 \Rightarrow \underline{\underline{H}} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$\det \underline{\underline{H}} = 4 - 1 = 3 > 0 \Rightarrow$ elégséges feltétel ✓

$h_{11} > 0 \Rightarrow$ pozitív definit $\Rightarrow$ lokális minimum

4. feladat

$$f(x;y) = x^2 + xy + y^2 + 8/x + 8/y$$

$$\partial_x f = 2x + y - 8x^{-2} = 0 \quad (1)$$
$$\partial_y f = 2y + x - 8y^{-2} = 0 \quad (2)$$

1db kell: $! \; x = y \Rightarrow 3x - \frac{8}{x^2} = 0$

$$x^3 = \frac{8}{3}$$

$$x = y = \frac{2}{\sqrt[3]{3}}$$

$$\partial_x^2 f = 2 + 16x^{-3} \Rightarrow 2 + 16 \cdot \frac{3}{8} = 8$$
$$\partial_y^2 f = 2 + 16y^{-3} \Rightarrow 2 + 16 \cdot \frac{3}{8} = 8$$
$$\partial_x \partial_y f = 1 \Rightarrow 1$$

$$\underline{\underline{H}} = \begin{bmatrix} 8 & 1 \\ 1 & 8 \end{bmatrix}$$

$\det \underline{\underline{H}} = 64 - 1 = 63 > 0$ ✓

$h_{11} = 8 > 0 \Rightarrow$ lok min.

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5. feladat

$f(x;y) = \cos x \cos y \cos(x+y)$ $\quad P_1(\pi/2; \pi/2)$ $\; P_2(0;0)$

$$\partial_x f = -\sin x \cos y \cos(x+y) - \cos x \cos y \sin(x+y)$$
$$= -\cos y \left(\sin x \cos(x+y) + \cos x \sin(x+y)\right) =$$
$$= -\cos y \sin(2x+y)$$

$$\partial_y f = -\cos x \sin(x+2y)$$

$\left.\begin{array}{l} \cos \pi/2 = 0 \Rightarrow P_1 \checkmark \\ \sin 0 = 0 \Rightarrow P_2 \checkmark \end{array}\right\}$ Lehetséges szélsőérték helyek

$$\partial_x^2 f = -2\cos y \cos(2x+y)$$
$$\partial_y^2 f = -2\cos x \cos(x+2y)$$
$$\partial_x\partial_y f = \sin y \sin(2x+y) - \cos y \cos(2x+y) =$$
$$= -\cos(2x+2y)$$

$$\underline{\underline{H}}|_{P_1} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \qquad \det \underline{\underline{H}}|_{P_1} = -1 < 0$$

$$\Downarrow$$

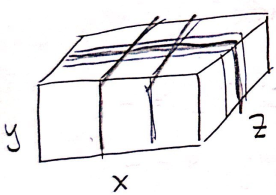
nincs sz.é.

$$\underline{\underline{H}}|_{P_2} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \qquad \det \underline{\underline{H}}|_{P_2} = 4-1 = 3 > 0$$

$$\Downarrow$$

$h_{11} > 0 \Rightarrow$ lok min.

6. feladat



$$\ell = 2x + 6y + 4z$$
$$V = 4{,}5 = xyz$$
$$z = \frac{4{,}5}{xy} \Rightarrow 4z = \frac{18}{xy}$$

$$\ell(x;y) = 2x + 6y + \frac{18}{xy} \qquad x;y > 0\,!$$

$$\left.\begin{aligned} \partial_x \ell &= 2 - 18x^{-2}y = 0 \Rightarrow 2x^2y = 18 \\ \partial_y \ell &= 6 - 18x^{-1}y^{-2} = 0 \Rightarrow 6xy^2 = 18 \end{aligned}\right\} \begin{aligned} 2x &= 6y \\ x &= 3y \end{aligned}$$

$x = 3$
$y = 1$
$z = 1{,}5$

$$\partial_x^2 \ell = 36x^{-3}y \Rightarrow 4/3$$
$$\partial_y^2 \ell = 36xy^{-3} \Rightarrow 12$$
$$\partial_x\partial_y \ell = 18x^{-2}y^{-2} \Rightarrow 2$$

$$\underline{\underline{H}} = \begin{bmatrix} 4/3 & 2 \\ 2 & 12 \end{bmatrix}$$

$\det \underline{\underline{H}} > 0$ ✓

$h_{11} > 0 \Rightarrow$ minimum

7. feladat

$$\ell = 4x + 4y + 4z \Rightarrow z = \frac{\ell}{4} - x - y$$

$$V = xyz \Rightarrow V(x;y) = xy\left(\frac{\ell}{4} - x - y\right)$$

$$\left.\begin{aligned} \partial_x V &= y(\ldots) - xy = 0 \\ \partial_y V &= x(\ldots) - xy = 0 \end{aligned}\right\} \quad x = y \Rightarrow (\ldots) = x$$

$$\frac{\ell}{4} - 2x = x$$

$$x = y = \ell/12 \qquad z = \ell/12$$

$$\partial_x^2 V = -2y \Rightarrow -\ell/6$$

$$\partial_y^2 V = -2x \Rightarrow -\ell/6$$

$$\partial_x \partial_y V = \frac{\ell}{4} - x - 2y \Rightarrow -\ell/12$$

$$\underline{\underline{H}} = \begin{bmatrix} -1/6 & -1/12 \\ -1/12 & -1/6 \end{bmatrix} \ell$$

$$\det \underline{\underline{H}} = \ell^2 \left(1/36 - 1/144\right) > 0$$

$$h_{11} < 0 \Rightarrow \text{maximum.}$$

8. feladat

$$f(x;y) = x^2 y^2 \qquad x^2 + y^2 = 1$$

$$F(x;y) = x^2 y^2 + \lambda(x^2 + y^2 - 1)$$

$$\left.\begin{aligned} \partial_x F &= 2x(y^2 + \lambda) = 0 \\ \partial_y F &= 2y(x^2 + \lambda) = 0 \\ \partial_\lambda F &= x^2 + y^2 - 1 = 0 \end{aligned}\right\} \quad \begin{aligned} &x = 0 \quad y = \pm 1 \\ &x = \pm 1 \quad y = 0 \\ &x = y = \pm \frac{1}{\sqrt{2}} \end{aligned}$$

Maximumok: $f = 1/4$ $\quad (0;\pm 1) \quad (\pm 1; 0)$

Minimumok: $f = 0$ $\quad \left(\pm \frac{1}{\sqrt{2}}; \pm \frac{1}{\sqrt{2}}\right)$

Másik mo.:

$$\left.\begin{aligned} x &= \cos t \\ y &= \sin t \end{aligned}\right\} \quad f(t) = \cos^2 t \sin^2 t$$

$$f'(t) = 2\cos t(-\sin t)\sin^2 t + \cos^2 t \, 2\sin t \cos t$$

$$= 2\cos t \sin t(\cos^2 t - \sin^2 t) = 0$$

⋮

9. feladat

$$f(x;y) = x^2 + y^2 + xy - x \qquad x^2 + y^2 \leq 1$$

Globális:

$$\left.\begin{aligned} \partial_x f &= 2x + y - 1 = 0 \\ \partial_y f &= 2y + x = 0 \end{aligned}\right\} \quad \begin{aligned} x &= 2/3 \\ y &= -1/3 \end{aligned}$$

$$\left.\begin{aligned} \partial_x^2 f &= 2 \\ \partial_y^2 f &= 2 \\ \partial_{xy}^2 f &= 1 \end{aligned}\right\} \quad \underline{\underline{H}} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \det \underline{\underline{H}} = 4 - 1 = 3 > 0$$

$$h_{11} > 0 \Rightarrow \text{minimum}$$

$$f(2/3; -1/3) = -1/3$$

Feltételes:

$$F = x^2 + y^2 + xy - x + \lambda(x^2 + y^2 - 1)$$

$$\begin{aligned} \partial_x F &= 2x + y - 1 + 2\lambda x = 0 \\ \partial_y F &= 2y + x + 2\lambda y = 0 \\ \partial_\lambda F &= x^2 + y^2 - 1 = 0 \end{aligned} \quad \left| \begin{aligned} &x = 0 \quad y = 1 \Rightarrow f = 1 \\ &x = \sqrt{3}/2 \quad y = -1/2 \Rightarrow f \simeq -0{,}299 \\ &x = -\sqrt{3}/2 \quad y = -1/2 \Rightarrow f \simeq 2{,}299 \end{aligned} \right.$$

Max: $x = \sqrt{3}/2 \quad y = -1/2$

Min: $x = 2/3 \quad y = -1/3$