Matematika G2 - GY10

1. feladat

$$f(x;y)=\begin{cases}\dfrac{2xy}{x^2+y^2} & x^2+y^2>0\\ 0 & x^2+y^2=0\end{cases}$$

⌜Átviteli elv.: $f:\mathbb{R}^n\to\mathbb{R}^k$, $\lim\limits_{\underline{x}\to\underline{a}} f(\underline{x})=\underline{A}$, ha $\forall \underline{x}_n\to\underline{a}$: $f(\underline{x}_n)\to\underline{A}$ ⌟

- ! $\underline{x}_n=(1/n;1/n) \Rightarrow \lim\limits_{n\to\infty}\dfrac{2\cdot\frac{1}{n}\cdot\frac{1}{n}}{\frac{1}{n^2}+\frac{1}{n^2}}=\dfrac{2}{2}=1$
- ! $\underline{x}_n=(2/n;4/n) \Rightarrow \lim\limits_{n\to\infty}\dfrac{2\cdot 2/n\cdot 4/n}{4/n^2+16/n^2}=\dfrac{16}{20}$

$\not\exists$ ↯

$$g(x;y)=\frac{x-y}{x+y}$$

⌜Különböző görbe mentén odatartás⌟

$x=y$ egyenes: $\lim\limits_{x\to 0}\dfrac{x-x}{x+x}=\lim\limits_{x\to 0}\dfrac{0}{2x}=\dfrac{0}{2}=0$

$y=0$ egyenes: $\lim\limits_{x\to 0}\dfrac{x}{x}=1$

$\not\exists$ ↯

2. feladat

$$\lim_{\substack{x\to 3\\ y\to\infty}}\frac{xy-1}{y+1} \Rightarrow y=\frac{1}{x-3}\text{ , ha } x\to 3^+:\ y\to\infty$$

$$\lim_{x\to 3^+}\frac{x\frac{1}{x-3}-1}{\frac{1}{x-3}+1}=\lim_{x\to 3^+}\frac{x-x+3}{1+x-3}=\frac{3}{1}=3$$

Ell.: $\left|\dfrac{xy-1}{y+1}-3\right|=\left|\dfrac{xy-1-3y-3}{y+1}\right|=\left|\dfrac{(x-3)y-4}{y+1}\right|<\left|\dfrac{(x-3)y}{y}\right|=\underbrace{|x-3|}_{\delta}<\varepsilon$ ✓

$$\lim_{\underline{x}\to\underline{0}} \frac{x+y+2z}{x-z+xy}$$

- $$\lim_{z\to 0}\lim_{x\to 0}\lim_{y\to 0} \frac{x+y+2z}{x-z+xy} = \lim_{z\to 0} \frac{2z}{-z} = -2$$

$\neq$

- $$\lim_{x\to 0}\lim_{y\to 0}\lim_{z\to 0} \frac{x+y+2z}{x-z+xy} = \lim_{x\to 0} \frac{x}{x} = 1$$

⌜Ha konvergens, az origóba tartás sorrendje mindegy lenne.⌟

3. feladat

$$f(x;y) = \frac{x^2y^2}{x^2+y^2} \qquad \underline{x}_0 = \underline{0} \qquad \begin{aligned} x &= r_n \cos\varphi_n \\ y &= r_n \sin\varphi_n \end{aligned}$$

$$\lim_{r_n\to 0} \frac{r_n^4 \cos^2\varphi_n \sin^2\varphi_n}{r_n^2} = 0 \;\checkmark$$

$$g(x;y) = \frac{x^3y}{x^6+y^2} \qquad \underline{x}_0 = \underline{0} \qquad y = mx^k \qquad ! \; y = mx^3$$

$$\lim_{x\to 0} \frac{mx^6}{x^6+m^2x^6} = \lim_{x\to 0} \frac{m}{1+m^2} \;\lightning$$

4. feladat

a, $$\lim_{\substack{x\to\infty \\ y\to\infty}} \frac{x+y}{x^2-xy+y^2} = \lim \frac{\frac{x}{xy}+\frac{y}{xy}}{\frac{x^2}{xy}-1+\frac{y^2}{xy}} = \lim \frac{\overset{\to 0}{1/y}+\overset{\to 0}{1/x}}{\underbrace{x/y-1+y/x}_{\geq 2}} = 0$$

b, $$\lim_{\substack{x\to 0 \\ y\to\infty}} x\cos^2 y \sim \text{rendőr elv} \Rightarrow 0$$

c, $\lim\limits_{\substack{x \to 0 \\ y \to 0}} \frac{x^2+y^2}{\sqrt{x^2+y^2+4}-2}$

$\lim \frac{(x^2+y^2)(\sqrt{x^2+y^2+4}+2)}{x^2+y^2+4-4} = \lim \sqrt{x^2+y^2+4}+2 = 4$

d, $\lim\limits_{\substack{x \to \infty \\ y \to 0}} \left(1+1/x\right)^{\frac{x^2}{x+y}} = \lim \left(\left(1+1/x\right)^x\right)^{\frac{x}{x+y}} = e$

5. feladat

$f(x;y) = x^2-2xy-4y^2 \qquad P(1;-1) \quad \underline{v}(1;-1)$

$\ulcorner \lim\limits_{\lambda \to 0} \frac{f(\underline{x}+\lambda\underline{v})-f(\underline{x})}{\lambda} \sim$ iránymenti derivált $\lrcorner$

$\lim\limits_{\lambda \to 0} \frac{(x+\lambda)^2-2(x+\lambda)(y-\lambda)-4(y-\lambda)^2-x^2+2xy+4y^2}{\lambda}$

$= \ldots = 2x+2x-2y+8y = 4x+6y = \frac{\partial f}{\partial v}$

$\left.\frac{\partial f}{\partial v}\right|_P = 4-6 = -2$

6. feladat

$$f(x;y) = x^3 - 5x^2y + 3xy^2 - 12y^3 + 5x - 6y + 7$$

$$\frac{\partial f}{\partial x} = 3x^2 - 10xy + 3y^2 + 5$$

$$\frac{\partial f}{\partial y} = -5x^2 + 6xy - 36y^2 - 6$$

$$\frac{\partial^2 f}{\partial x^2} = 6x - 10 \qquad \frac{\partial^2 f}{\partial x \partial y} = -10x + 6y$$

$$\frac{\partial^2 f}{\partial y^2} = 6x - 72y \qquad \frac{\partial^2 f}{\partial y \partial x} = -10x + 6y$$

✓

$$g(x;y) = x^y$$

$$\frac{\partial f}{\partial x} = y x^{y-1} \qquad \frac{\partial f}{\partial y} = x^y \ln x$$

$$\frac{\partial^2 f}{\partial x^2} = y(y-1)x^{y-2} \qquad \frac{\partial^2 f}{\partial y^2} = x^y \ln^2 x$$

$$\frac{\partial^2 f}{\partial x \partial y} = x^{y-1}(1 + y\ln x) \qquad \frac{\partial^2 f}{\partial y \partial x} = yx^{y-1}\ln x + x^y \frac{1}{x}$$

$$h(x) = e^{x^2y} - 2x^2y^3 \sin(\ln x + y)$$

$$\frac{\partial h}{\partial x} = 2xye^{x^2y} - 4xy^3 \sin(\dots) - 2x^2y^3 \cos(\dots)\frac{1}{x}$$

$$\frac{\partial h}{\partial y} = x^2e^{x^2y} - 6x^2y^2 \sin(\dots) - 2x^2y^3 \cos(\dots)$$

7. feladat

$f(x;y) = x\ln(x+y)$

$P(-2;3)$ grad $f|_P = ?$

$$\frac{\partial f}{\partial x} = \ln(x+y) + \frac{x}{x+y} \xrightarrow{P} \ln 1 + \frac{-2}{1} = 0-2=-2$$

$$\frac{\partial f}{\partial y} = \frac{x}{x+y} \xrightarrow{P} -2$$

$$\text{grad } f|_P = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

$g(x;y;z) = z - \sqrt{x^2+y^2}$

$P(3;-4;7)$ grad $g|_P = ?$

$$\frac{\partial f}{\partial x} = \frac{-1}{2}(x^2+y^2)^{-1/2} \cdot 2x \Rightarrow \frac{-1}{5} \cdot 3 = -3/5$$

$$\frac{\partial f}{\partial y} = -\frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2y \Rightarrow \frac{-1}{5}(-4) = 4/5$$

$$\frac{\partial f}{\partial z} = 1$$

$$\text{grad } g|_P = \begin{bmatrix} -3/5 \\ 4/5 \\ 1 \end{bmatrix}$$

8. feladat

$$f(x;y) = 3x^2 - 4xy + 2x + y^2 + 1$$

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 6x - 4y + 2 = 0 \\ \frac{\partial f}{\partial y} &= -4x + 2y = 0 \end{aligned}\right\}$$

$$-2x + 2 = 0 \qquad \begin{aligned} x &= +1 \\ x &= +2 \end{aligned} \quad \Rightarrow \quad P = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$