Matematika G2 - GY7 Sándor Tibor

1. feladat

$\ulcorner \sqrt[n]{a} \to 1 \lrcorner$

a) $\sum_{n=1}^{\infty} \sqrt[n]{x}$ $\qquad \mathcal{D} = \mathbb{R}^+$

ha $x = 0$ $\quad \sqrt[n]{x} \to 0 \to \sum 0 \sim s(x) = 0$
ha $x \neq 0$ $\quad \sqrt[n]{x} \to 1 \to \sum 1 \sim$ divergens

$K = \{0\}$

$\mathcal{D} = \mathbb{R} \setminus \{-1\}$

b) $\sum_{n=1}^{\infty} \left(\frac{x-1}{x+1}\right)^n \sim \sum q^n$ konv $\Leftrightarrow |q| < 1$

$\left|\frac{x-1}{x+1}\right| < 1 \Rightarrow |x-1| < |x+1| \Rightarrow K : (0; +\infty)$

$$s(x) = \frac{a_0}{1-q} = \frac{\frac{x-1}{x+1}}{1 - \frac{x-1}{x+1}} = \frac{x-1}{x+1-x+1} = \frac{x-1}{2}$$

$\mathcal{D} = \mathbb{R}$

c) $\sum_{n=0}^{\infty} \sin^{2n} x = \sum_{n=1}^{\infty} (\sin^2 x)^n \Rightarrow |\sin^2 x| < 1$

$\Downarrow$

$s(x) = \frac{1}{1-\sin^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$ | $K = \{x \in \mathbb{R} \mid x \neq \pi/2 + k\pi\}$

2. feladat

a) $\sum_{n=1}^{\infty} \frac{1}{1+x^{2n}}$ ⌜Majorálás⌟

$$\leq$$

$$\sum \frac{1}{x^{2n}} = \sum \left(\frac{1}{x^2}\right)^n \Rightarrow \begin{matrix} |x|>1 \text{ absz. konv.} \\ |x|<1 \text{ div.} \end{matrix}$$

ha $x^2 = 1 \Rightarrow \sum \frac{1}{2} \Rightarrow$ div

b) $\sum_{n=1}^{\infty} (-1)^n n^{-x}$ ⌜Leibnitz-sor⌟

konvergens, ha n^{-x} mon. cs. nullsorozat

$$\lim_{n\to\infty} \frac{1}{n^x} \sim \begin{matrix} \text{ha } x>0 \sim \text{feltételesen k.} \\ \text{ha } x>1 \sim \text{absz k.} \end{matrix}$$

c) $\sum_{n=1}^{\infty} \frac{\left|\frac{z-i-1}{3}\right|^n}{z}$ ⌜Gyökteszt⌟

$$\lim_{n\to\infty} \sqrt[n]{\frac{\left|\frac{z-i-1}{3}\right|^n}{z}} = \left|\frac{z-i-1}{3}\right| = q$$

Ha $q<1 \sim |z-(i+1)|<3 \Rightarrow$ abszolút konvergens

d) $\sum_{n=1}^{\infty} \frac{\cos(3x^3+(\pi/3)nx^2)}{3^n+x^4n^4} \leq \sum \left(\frac{1}{3}\right)^n$

$$\lim_{x\to 0} \sum_{n=0}^{\infty}(\dots) = \sum_{n=0}^{\infty} \lim_{x\to 0} \frac{\cos(\dots)}{3^n+x^4n^4} = \sum \left(\frac{1}{3}\right)^n = \frac{1}{1-1/3} = \frac{3}{2}$$

↑ abszolút konvergencia

3. feladat

$$\int_0^2 \sum_{n=1}^{\infty} \frac{x^n}{e^{nx}}\,dx \overset{?}{=} \sum_{n=1}^{\infty} \int_0^2 \frac{x^n}{e^{nx}}\,dx \quad \text{Igen!}$$

feltétel: egyenletes konvergencia a $(0;2)$-n, valamint (f_n) folytonos ✓

$$\sum_{n=1}^{\infty} \left(\frac{x}{e^x}\right)^n < \sum \left(\frac{2}{e}\right)^n \checkmark$$

4. feladat

$$\sum_{n=1}^{\infty} \frac{\arctan(x/n)}{n^2} \text{ tagonként deriválható?}$$

Igen!

f_n folytonos ✓

$$f_n' = \frac{1}{1+(x/n)^2} \cdot \frac{1}{n^2} \cdot \frac{1}{n} \quad \text{folytonos} \checkmark$$

$$\sum \frac{\arctan(x/n)}{n^2} < \sum \frac{\pi/2}{n^2} \quad \text{konvergens} \checkmark$$

$$\sum \frac{1/n^3}{1+(x/n)^2} < \sum \frac{1}{n^3} \quad \text{konvergens} \checkmark$$

5. feladat

a, $\sum_{n=1}^{\infty} n(x-2)^n \qquad x_0 = 2, \quad a_n = n$

$$\frac{1}{r} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{n+1}{n}\right| = 1 \qquad r = 1$$

$x = 2+1 = 3 \Rightarrow \sum n \Rightarrow \text{div}$

$x = 2-1 = 1 \Rightarrow \sum n(-1)^n \Rightarrow \text{div}$

$$K = (+1; +3)$$

b) $\sum_{n=2}^{\infty} \frac{x^n}{2^n(n-1)}$ $\quad x_0 = 0$

$a_n = \frac{1}{2^n(n-1)}$

$\frac{1}{r} = \lim \left| \frac{2^n(n-1)}{2^{n+1} n} \right| = \frac{1}{2} \quad r = 2$

$x = -2 \Rightarrow \sum \frac{(-2)^n}{2^n(n-1)} = \sum (-1)^n \frac{1}{n-1} \Rightarrow \text{konv}$

$x = 2 \Rightarrow \sum \frac{1}{n-1} \Rightarrow \text{div}$

$K = [-2;2)$

c) $\sum_{n=1}^{\infty} (-1)^n \frac{(x-2)^n}{3}$ $\quad x_0 = 2$

$a_n = \frac{(-1)^n}{3}$

$\frac{1}{r} = \lim \left| \frac{(-1)^{n+1}}{(-1)^n} \right| = 1 \Rightarrow r = 1$

$x = 2-1 = 1 \quad \sum \frac{(-1)^{2n}}{3} \Rightarrow \text{div}$

$x = 2+1 = 3 \quad \sum (-1)^n \frac{3^n}{3} \Rightarrow \text{div}$

$K = (1;3)$

d) $\sum_{n=1}^{\infty} \left(4 - \frac{1}{n}\right)^n x^n$ $\quad x_0 = 0$

$a_n = \left(4 - \frac{1}{n}\right)^n$

$\frac{1}{r} = \lim \left| 4 - \frac{1}{n} \right| = 4 \quad r = \frac{1}{4}$

$x = -1/4 \quad \sum \left(4 - \frac{1}{n}\right)^n \left(-\frac{1}{4}\right)^n = \sum (-1)^n \left(1 - \frac{1}{4n}\right)^n \Rightarrow \text{div}$

$x = 1/4 \quad \sum \left(4 - \frac{1}{n}\right)^n \left(\frac{1}{4}\right)^n = \sum 1^n - \left(\frac{1}{4n}\right)^n \Rightarrow \text{div}$

$K = (-1/4; 1/4)$

e, $\sum_{n=1}^{\infty}\left(\frac{4n(-1)^n+n+2}{2n}\right)^n x^n \quad x_0=0$

$\overline{\lim}\left|\frac{4n(-1)^n+n+2}{2n}\right|$ ↗ n páros $5/2$ ↘ n páratlan $3/2$

$K=(-2/5;2/5)$

6. feladat

a, $\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}z^n \qquad z_0=0$

$\lim\left|\frac{((n+1)!)^2}{(n!)^2}\cdot\frac{(2n)!}{(2n+2)!}\right|=\lim\frac{(n+1)^2}{(2n+1)(2n+2)}=\frac{1}{4} \qquad r=4$

b, $\sum_{n=1}^{\infty}\frac{x^{2n-1}}{2n-1}=\int\sum_{n=1}^{\infty}x^{2n-2}\,dx=$

$=\int\sum_{n=1}^{\infty}(x^{n-1})^2\,dx=\int\sum_{n=0}^{\infty}|x^2|^n$

$|x^2|<1 \Rightarrow$ konvergens $\qquad K=(-1;1)$

$\sum_{n=0}^{\infty}(x^2)^n=\frac{1}{1-x^2}$

$s(x)=\int_0^x\frac{1}{1-t^2}dt=\int_0^x\frac{1/2}{1-t}+\frac{1/2}{1+t}\,dt=$

$=\frac{1}{2}[\ln(x+1)+\ln(x-1)]$

c) $\sum_{n=0}^{\infty} (n+1)(n+2)\, x^n = \sum_{n=0}^{\infty} (x^{n+2})''$

$\sum_{n=0}^{\infty} x^{n+2} = \frac{x^2}{1-x}$ $\quad K = (-1;1)$

$s(x) = \left(\frac{x^2}{1-x}\right)'' = \frac{2}{(1-x)^3}$