Matematika G1 - GY6 Sándor Tibor

1. feladat

a, $\sum_{n=1}^{\infty} \frac{[\cos^n(\pi/2)]^{4n}}{n^n + 1}$

⌜Majorálás: $a_n < b_n$ és $\sum b_n$ konvergens, akkor $\sum a_n$ is az⌟

$\cos x \in [-1;1]$

$$\sum \frac{(\cos^n \pi/2)^{4n}}{n^n + 1} \leq \sum \frac{1^{4n}}{n^n+1} < \sum \frac{1}{n^n} < \sum \frac{1}{n^2}$$

↳ konvergens

b, $\sum_{n=1}^{\infty} \frac{2n^2}{(2+1/n)^n}$

⌜Gyökteszt: $\lim \sqrt[n]{|a_n|} = q$ ha $|q|<1 \Rightarrow \sum a_n$ konv.⌟

$$\lim_{n\to\infty} \frac{\sqrt[n]{2n^2}}{2+\frac{1}{n}} = \frac{1}{2} < 1 \Rightarrow \text{konvergens}$$

c, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left(1-\frac{1}{n}\right)^n$

⌜Minorálás: $a_n > b_n$ és $\sum b_n$ divergens, akkor $\sum a_n$ is az⌟

$\vee$

$$\sum \frac{1}{n}\underbrace{\left(1-\frac{1}{n}\right)^n}_{\to \frac{1}{e}} < \sum \frac{1}{n}\cdot\frac{1}{e} = \frac{1}{e}\sum\frac{1}{n} \Rightarrow \text{divergens}$$

d, $\sum_{n=0}^{\infty} \frac{n!}{2^n+1}$ ⌜Hányadosteszt: ha $\lim\left|\frac{a_{n+1}}{a_n}\right|=q$ és $q<1$, akkor $\sum a_n$ konvergens⌟

$$\lim_{n\to\infty} \frac{(n+1)!}{2^{n+1}+1} \cdot \frac{2^n+1}{n!} = \lim_{n\to\infty} \frac{(n+1)(2^n+1)}{(2\cdot 2^n+1)} =$$

$$= \lim_{n\to\infty} \frac{(n+1)\left(1+\frac{1}{2^n}\right)}{2+\frac{1}{2^n}} = \frac{\infty}{2} = \infty \Rightarrow \text{divergens}$$

e, $\sum_{n=1}^{\infty} \frac{n(-1)^{n+1}}{n^2-1} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2-1}$

$\Downarrow$ Leibnitz-típusú ⌜$\sum (-1)^n a_n$ konvergens, ha $(a_n) \searrow \to 0$ ⌟

$$\lim_{n\to\infty} \frac{n}{n^2-1} = 0 \ \checkmark \Rightarrow \text{konvergens}$$

f, $\sum_{n=1}^{\infty} \frac{n}{e^n}$ ⌜Integrálkritérium: $\sum |f_n|$ konvergens, ha $\int_1^{\infty} f(x)dx$ konvergens⌟

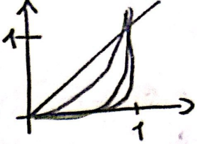
$$f(x) = \frac{x}{e^x} \Rightarrow \int_1^{\infty} \frac{x}{e^x}\,dx = \int xe^{-x}dx =$$

$f=x \quad g=-e^{-x}$
$f'=1 \quad g'=e^{-x}$

$$= -xe^{-x}\Big|_1^{\infty} + \int_1^{\infty} e^{-x}dx = -xe^{-x} - e^{-x}\Big|_1^{\infty} = 0-\left(-\frac{1}{e}-\frac{1}{e}\right) = \frac{2}{e}$$

$\Downarrow$ konvergens

2. feladat



a, $f_n = x^n$

$\mathcal{D} = \mathbb{R}$

K: a^n konv, ha $\begin{matrix} |a|<1 & (\to 0) \\ a=1 & (\to 1) \end{matrix} \Rightarrow K = (-1;1]$

$f = \begin{cases} 0, \text{ha } x \in (-1;1) \\ 1, \text{ha } x = 1 \end{cases}$

b, $f_n = \dfrac{x^{n+2}+1}{x^n}$

$\mathcal{D}_f = \mathbb{R} \setminus \{0\}$

K: $\lim\limits_{n\to\infty} f_n = \lim\limits_{n\to\infty} \dfrac{1 + \frac{1}{x^{n+2}}}{\frac{1}{x^2}}$ $\quad \frac{1}{x^{n+2}} = \left(\frac{1}{x}\right)^{n+2}$

$\frac{1}{x^2} \sim$ korlátos (nem függ n-től)

$\left.\begin{matrix} |x| < 1 \to \text{div} \\ x = 1 \to 2 \\ x = -1 \to \text{div} \\ |x| > 1 \to x^2 \end{matrix}\right\}$ $K = \mathbb{R} \setminus [-1;1)$

$f = \begin{cases} x^2, \text{ha } x \in (-\infty;-1) \cup (1;\infty) \\ 2, \text{ha } x = 1 \end{cases}$

c, $f_n = \dfrac{\sin nx}{n}$

$\mathcal{D}_f = \mathbb{R}$

K: Rendőr elv: $-\frac{1}{n} \leq \frac{\sin nx}{n} \leq \frac{1}{n} \Rightarrow 0$

$K = \mathbb{R} \quad f = 0$

d, $f_n = \ln^n x$

$\mathcal{D}_f = \mathbb{R}^+$

K: $\ln x \in (-1;1] \Rightarrow K = (1/e; e]$

$f = \begin{cases} 0, \text{ha } x \in (1/e; e) \\ 1, \text{ha } x = e \end{cases}$

e, $f_n = n \cdot \sin\left(\frac{x}{n}\right)$

$\mathcal{D} = \mathbb{R}$

$$K: \lim_{n\to\infty} f_n = \lim_{n\to\infty} \cancel{n} \underbrace{\frac{\sin\frac{x}{n}}{x/n}}_{\to 1} x/\cancel{n} = x$$

$K = \mathbb{R} \quad f = x$

3. feladat

$$f_n = \frac{2x^3n^2}{x^2n^2+5} \qquad \mathcal{D} = \mathbb{R}$$

$$\lim_{n\to\infty} f_n = \frac{2x^3}{x^2+5/n^2} = 2x$$

$$|f_n(x) - f(x)| \overset{?}{<} \varepsilon$$

$$\left|\frac{2x^3n^2}{x^2n^2+5} - 2x\right| = \left|\frac{2x^3n^2 - 2x^3n^2 - 10x}{x^2n^2+5}\right| =$$

$$x \in [2;5]$$

$$= \left|\frac{-10x}{x^2n^2+5}\right| = \frac{10x}{x^2n^2+5} \leq \frac{50}{4n^2+5} \leq \frac{50}{4n^2} < \varepsilon$$

$n > \sqrt{\frac{50}{4\varepsilon}}$ => egyenletesen konv.

4. feladat

$$\lim_{n\to\infty} \int_0^{2\pi} \frac{\sin(n^4x^2+3)}{x^2+n^3} \overset{?}{=} 0$$

(f_n) folytonos ✓

(f_n) egyenletesen konvergens ?

$$f(x) = \lim_{n\to\infty} \frac{\overbrace{\sin(\dots)}^{\in[-1;1]}}{\underbrace{x^2+n^3}_{\to\infty}} = 0$$

$$|f_n - f| = \left|\frac{\sin(\dots)}{x^2+n^3}\right| \le \frac{1}{n^3} < \varepsilon \qquad n > \varepsilon^{-1/3}$$ ✓

$$\int_0^{2\pi} 0\,dx = 0$$ ✓

5. feladat

$$f_n(x) = x^2 + \frac{1}{n}\sin\left[n(x+\pi/2)\right]$$

$$\lim_{n\to\infty} f_n = x^2$$

$$f_n'(x) = 2x + \frac{1}{n}n\cos(\dots) = 2x + \underbrace{\cos(\dots)}_{\text{nem konvergens}}$$

$\nexists f'$