Matematika G2 - GY5

1. feladat

$$\underline{\underline{A}} = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} \qquad \det(\underline{\underline{A}} - \lambda \underline{\underline{E}}) = \begin{vmatrix} 4-\lambda & 3 \\ 1 & 2-\lambda \end{vmatrix} =$$

$$= (4-\lambda)(2-\lambda) - 3 \cdot 1 = \lambda^2 - 6\lambda + 5 = (\lambda-1)(\lambda-5) = 0$$

Sajátértékek: $\lambda_1 = 1 \quad \lambda_2 = 5$

Sajátvektorok: $(\underline{\underline{A}} - \lambda \underline{\underline{E}})\underline{v} = 0$

- $\lambda_1 = 1$: $\begin{bmatrix} 4-1 & 3 \\ 1 & 2-1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix}$

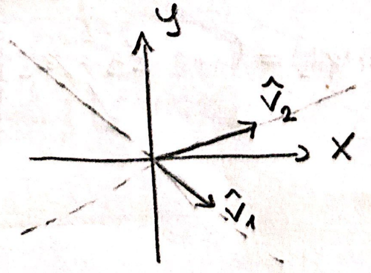
  $x_1 = -y_1 \Rightarrow \underline{v}_1 = t_1 [1 \;\; -1]^T$

- $\lambda_2 = 5$: $\begin{bmatrix} 4-5 & 3 \\ 1 & 2-5 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix}$

  $x_2 = 3y_2 \Rightarrow \underline{v}_2 = t_2 [3 \;\; 1]^T$

Egységhosszú, első koordináta $\oplus$

$\lambda_1 = 1 \quad \hat{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \qquad \lambda_2 = 5 \quad \hat{v}_2 = \begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$



$$\underline{\underline{B}} = \begin{bmatrix} -2 & -8 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix} \qquad \det(\underline{\underline{B}} - \lambda \underline{\underline{E}}) = \begin{vmatrix} -2-\lambda & -8 & -12 \\ 1 & 4-\lambda & 4 \\ 0 & 0 & 1-\lambda \end{vmatrix} =$$

$$= (1-\lambda)\left[(-2-\lambda)(4-\lambda) - 1(-8)\right] = (1-\lambda)\left[\lambda^2 - 2\lambda - 8 + 8\right] =$$

$$= (1-\lambda)(\lambda-2)\lambda = 0 \Rightarrow \lambda_1 = 0 \quad \lambda_2 = 1 \quad \lambda_3 = 2$$

- $\lambda_1 = 0: \begin{bmatrix} -2 & -8 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow z_1 = 0 \quad x_1 + 4y_1 = 0$

  $\underline{v}_1 = t_1 \begin{bmatrix} 4 & -1 & 0 \end{bmatrix}^T$

- $\lambda_2 = 1: \begin{bmatrix} -3 & -8 & -12 \\ 1 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 1 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow y_2 = 0 \quad x_2 = -4z_2$

  $\underline{v}_2 = t_2 \begin{bmatrix} 4 & 0 & -1 \end{bmatrix}^T$

- $\lambda_3 = 2: \begin{bmatrix} -4 & -8 & -12 \\ 1 & 2 & 4 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow z_3 = 0 \quad x_3 = -2y_3$

  $\underline{v}_3 = t_3 \begin{bmatrix} 2 & -1 & 0 \end{bmatrix}^T$

$$\underline{\underline{C}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \det(\underline{\underline{C}} - \lambda \underline{\underline{E}}) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$$

$$\lambda_1 = +i \quad \lambda_2 = -i$$

- $\lambda_1 = +i: \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \Rightarrow -ix_1 + y_1 = 0 \quad \underline{v}_1 = t_1 \begin{bmatrix} 1 \\ i \end{bmatrix}$

- $\lambda_2 = -i: \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \Rightarrow ix_2 + y_2 = 0 \quad \underline{v}_2 = t_2 \begin{bmatrix} 1 \\ -i \end{bmatrix}$

$$\underline{\underline{R}} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \quad \det(\underline{\underline{R}} - \lambda \underline{\underline{E}}) = \begin{vmatrix} c-\lambda & -s \\ s & c-\lambda \end{vmatrix} =$$

$$= (\cos\varphi - \lambda)^2 + \sin^2\varphi = 0$$

$$\cos^2\varphi - 2\lambda\cos\varphi + \lambda^2 + \sin^2\varphi = 0$$

$$\cancel{\cos^2}\varphi - 2\lambda\cos\varphi + \lambda^2 + 1 - \cancel{\cos^2}\varphi = 0$$

$$\lambda^2 - 2\lambda\cos\varphi + 1 = 0$$

$$\lambda_{12} = \frac{\cancel{2}\cos\varphi \pm \sqrt{\cancel{4}\cos^2\varphi - \cancel{4}}}{\cancel{2}} = \cos\varphi \pm i\sin\varphi = e^{\pm i\varphi}$$

- $\lambda_1 = e^{i\varphi}$: $\begin{bmatrix} -i\sin\varphi & -\sin\varphi \\ \sin\varphi & -i\sin\varphi \end{bmatrix} \Rightarrow \underline{v}_1 = t_1 \begin{bmatrix} 1 \\ -i \end{bmatrix}$

- $\lambda_2 = e^{-i\varphi}$: $\begin{bmatrix} i\sin\varphi & -\sin\varphi \\ \sin\varphi & i\sin\varphi \end{bmatrix} \Rightarrow \underline{v}_2 = t_2 \begin{bmatrix} 1 \\ i \end{bmatrix}$

Tesztelés:

$$\begin{array}{cc} & \begin{bmatrix} 1 \\ -i \end{bmatrix} \\ \begin{bmatrix} c & -s \\ s & c \end{bmatrix} & \begin{bmatrix} c+si \\ s-ci \end{bmatrix} \end{array} = (c+si)\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$\begin{array}{cc} & \begin{bmatrix} 1 \\ i \end{bmatrix} \\ \begin{bmatrix} c & -s \\ s & c \end{bmatrix} & \begin{bmatrix} c-si \\ s+ci \end{bmatrix} \end{array} = (c-si)\begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\underline{\underline{D}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$\lambda_{12} = 1 \rightarrow$ algebrai mult $= 2$

$\lambda_{34} = 2 \rightarrow$ algebrai mult $= 2$

• $\lambda_{12} = 1$: $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$rg = 2 \leftarrow$

geom mult $= 2$

$x_{13} = 0$

$x_{14} = 0$

$\left.\begin{matrix} x_{11} \\ x_{12} \end{matrix}\right\}$ tetszőleges

$\underline{v}_1 = t_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $\underline{v}_2 = t_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

• $\lambda_{34} = 2$: $\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$rg = 3$

geom mult $= 1$

$x_{21} = 0$

$x_{22} = 0$

$x_{24} = 0$

x_{23} tetszőleges

$\underline{v}_3 = t_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$$\underline{\underline{E}} = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

$\lambda_{123} = 4 \rightarrow$ algebrai mult $= 3$

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$y = 0$

$z = 0$

x tetszőleges

$\underline{v} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$rg = 2$, geom mult $= 1$

$$\underline{\underline{F}} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$\det(\underline{\underline{E}} - \lambda \underline{\underline{E}}) = \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix}$$

$\lambda_{12} = 1$ $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

$x + 2y + z = 0$

$y = y$ $z = z$

$x = -2y - z$

$\underline{v}_1 = t_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ $\underline{v}_2 = t_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

2. feladat

- z-körüli forgatás:

$$\begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \lambda_3 = 1 \quad \underline{v}_3 = t_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Többi: $\lambda_{12} = \cos\varphi \pm i\sin\varphi$

$$\underline{v}_1 = t_1 \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix} \qquad \underline{v}_2 = t_2 \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}$$

- xy-ra vetítés:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} \lambda_1 = 1 \\ \lambda_2 = 1 \\ \lambda_3 = 0 \end{matrix} \quad \begin{matrix} \underline{v}_1 = t_1 \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T \\ \underline{v}_2 = t_2 \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T \\ \underline{v}_3 = t_3 \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \end{matrix}$$

- xy-ra tükrözés:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \begin{matrix} \lambda_1 = 1 \\ \lambda_2 = 1 \\ \lambda_3 = -1 \end{matrix} \quad \begin{matrix} \underline{v}_1 = t_1 \hat{i} \\ \underline{v}_2 = t_2 \hat{j} \\ \underline{v}_3 = t_3 \hat{k} \end{matrix}$$

4. feladat

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 0 & 5 \end{bmatrix} \quad \Delta \Rightarrow \quad 1; 4; 5$$

$$\underline{\underline{B}} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 9 \end{bmatrix} \quad \Delta \Rightarrow \quad 5; 7; 9$$

5. feladat

$$\underline{\underline{B}} = \begin{bmatrix} -2 & -8 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix} \qquad \underline{\underline{B}}^{10} = ?$$

$\lambda_1 = 0$ $\quad \underline{v}_1 = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$

$\lambda_2 = 1$ $\quad \underline{v}_2 = \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}$

$\lambda_3 = 2$ $\quad \underline{v}_3 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$

$$\underline{\underline{T}} = \begin{bmatrix} 4 & 4 & 2 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\underline{\underline{T}}^{-1} = \begin{bmatrix} 1/2 & 1 & 2 \\ 0 & 0 & -1 \\ -1/2 & -2 & -2 \end{bmatrix}$$

$$\hat{\underline{\underline{B}}} = \underline{\underline{T}}^{-1} \underline{\underline{B}} \, \underline{\underline{T}} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\underline{\underline{B}}^{10} = \underline{\underline{T}} \, \hat{\underline{\underline{B}}}^{10} \, \underline{\underline{T}}^{-1} = \underline{\underline{T}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1024 \end{bmatrix} \underline{\underline{T}}^{-1} = \begin{bmatrix} -1024 & -4096 & -4100 \\ 512 & 2048 & 2048 \\ 0 & 0 & 1 \end{bmatrix}$$

6. feladat

$$e^{10\underline{\underline{B}}} = ?$$

$$e^{10\hat{\underline{\underline{B}}}} = \begin{bmatrix} e^{0} & 0 & 0 \\ 0 & e^{10} & 0 \\ 0 & 0 & e^{100} \end{bmatrix}$$

$$e^{10\underline{\underline{B}}} = \underline{\underline{T}} \, e^{10\hat{\underline{\underline{B}}}} \, \underline{\underline{T}}^{-1} = \dots$$

7. feladat

$$-3x^2+23y^2+26\sqrt{3}\,xy=144$$

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} -3 & 13\sqrt{3} \\ 13\sqrt{3} & 23 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=144$$

$\Rightarrow$ hiperbola

$\lambda_1=36$ $\quad \hat{v}_1=\begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}$ $\quad \hat{v}_2=\begin{bmatrix} -\sqrt{3}/2 \\ 1/2 \end{bmatrix}$

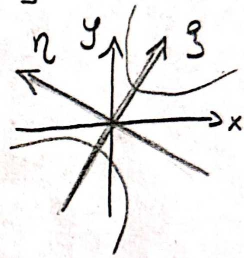
$\lambda_2=-16$

$$\underline{\underline{T}}=\begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}=\begin{bmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{bmatrix}$$

$$\underline{\underline{\Lambda}}=\begin{bmatrix} 36 & 0 \\ 0 & -16 \end{bmatrix}$$

$$36\xi^2-16\eta^2=144$$

$$\frac{\xi^2}{4}-\frac{\eta^2}{9}=1$$



$$57x^2+43y^2+14\sqrt{3}\,xy=576$$

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 57 & 7\sqrt{3} \\ 7\sqrt{3} & 43 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=576$$

$\lambda_1=64$

$\lambda_2=36$ $\Rightarrow$ ellipszis

$$2x^2-5=0$$

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=5$$

$\lambda_1=2$

$\lambda_2=0$ $\Rightarrow$ parabola