Matematika G1 - GY12 Sándor Tibor

1. feladat

$$\int \cos^3 x \sin x \, dx = \frac{-\cos^4 x}{4} + C$$

$$\sim \int f^{\alpha} f' = \frac{f^{\alpha+1}}{\alpha+1} + C$$

2. feladat

$$\int \cos^5 x \, dx = \int \cos^4 x \cos x \, dx = \int (\cos^2 x)^2 \cos x \, dx = \int (1-\sin^2 x)^2 \cos x \, dx =$$

$$= \int (1 - 2\sin^2 x + \sin^4 x) \cos x \, dx = \sin x - \frac{2\sin^3 x}{3} + \frac{\sin^5 x}{5} + C$$

3. feladat

$$\int \sin^4 x \cos^2 x \, dx = \int \sin^4 x (1-\sin^2 x) \, dx = \underbrace{\int \sin^4 x \, dx}_{1.} - \underbrace{\int \sin^6 x \, dx}_{2.}$$

1. $$\int \sin^4 x \, dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx = \frac{1}{4} \int 1 - 2\cos 2x + \cos^2 2x \, dx =$$

$$= \frac{1}{4} \int 1 - 2\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 4x \, dx = \frac{x}{4} - \frac{\sin 2x}{4} + \frac{x}{8} + \frac{\sin 4x}{32} + C$$

2. $$\int \sin^6 x \, dx = \frac{1}{8} \int (1-\cos 2x)^3 dx = \frac{1}{8} \int 1 - 3\cos 2x + 3\cos^2 2x - \cos^3 2x \, dx =$$

$$= \frac{1}{8} \int 1 - 3\cos 2x + 3\,\frac{1+\cos 4x}{2} - \cos 2x (1-\sin^2 2x) \, dx =$$

$$= \frac{1}{8}\int 1 - 3\cos 2x + \frac{3}{2} + \frac{3\cos 4x}{2} - \cos 2x + \cos 2x \sin^2 2x \, dx =$$

$$= \frac{1}{8}\left(\frac{5x}{2} - \frac{3\sin 2x}{2} + \frac{3\sin 4x}{8} - \frac{\sin 2x}{2} + \frac{\sin^3 2x}{6}\right) + C$$

4. feladat

$$\int \sin\sqrt{x}\, dx \qquad t = \sqrt{x} \qquad x = t^2 \qquad \frac{dx}{dt} = 2t \qquad dx = 2t\,dt$$

$$u \int \sin t \; 2t \, dt = 2\int t \cdot \sin t = -2t\cos t + 2\int \cos t \, dt =$$

$$f = t \qquad g' = \sin t$$
$$f' = 1 \qquad g = -\cos t$$

$$= -2t\cos t + 2\sin t + C = -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C$$

5. feladat

$$\int \frac{\ln \ln x}{x}\, dx \qquad t = \ln x \qquad dt/dx = \frac{1}{x} \qquad dx = x\,dt$$

$$\int \frac{\ln t}{x} x\,dt = \int \ln t \, dt = t\ln t - t + C = \ln x \ln\ln x - \ln x + C$$

$$\ulcorner \int \ln t \, dt = t\ln t - \int dt = t\ln t - t + C$$

$$f = \ln t \qquad g' = 1$$
$$f' = \frac{1}{t} \qquad g = t \qquad \lrcorner$$

6. feladat

$$\int |x|\,dx = |x|x - \int |x|\,dx \rightarrow \int |x|\,dx = \frac{x|x|}{2} + C$$

$f = |x| \qquad g' = 1$

$f' = \frac{|x|}{x} \qquad g = x$

7. feladat

$$\int \frac{\ln x + 1}{x^x - 1}\,dx \qquad t = x^x = e^{x\ln x} \qquad \frac{dt}{dx} = x^x(\ln x + 1) = t(\ln x + 1)$$

$$dx = \frac{dt}{t(\ln x + 1)}$$

$$\int \frac{\ln x + 1}{t - 1} \frac{dt}{t(\ln x + 1)} = \int \frac{dt}{t(t-1)}\,dt = \int \frac{A}{t} + \frac{B}{t-1}\,dt$$

$$\begin{cases} A + B = 0 \\ -A = 1 \end{cases} \qquad \begin{matrix} A = -1 \\ B = +1 \end{matrix}$$

$$\int \frac{-1}{t} + \frac{+1}{t-1}\,dt = -\ln|t| + \ln|t-1| + C = -\ln|x^x| + \ln|x^x - 1| + C$$

8. feladat

$$\int (x^2 - 3x + 2)\sqrt{2x-1} \qquad t = 2x - 1 \qquad \frac{dt}{dx} = 2 \quad dx = \frac{dt}{2}$$

$$x = \frac{t+1}{2} \qquad x^2 = \frac{(t+1)^2}{4} = \frac{t^2 + 2t + 1}{4}$$

$$\int\left(\frac{t^2}{4}+\frac{t}{2}+\frac{1}{4}-\frac{3t}{2}-\frac{3}{2}+2\right)\sqrt{t}\,\frac{dt}{2} =$$

$$= \int \frac{t^{5/2}}{8}-\frac{t^{3/2}}{2}+\frac{3t^{1/2}}{8}\,dt = \frac{t^{7/2}}{28}-\frac{t^{5/2}}{5}+\frac{t^{3/2}}{4}+C = \dots \qquad (t=2x+1)$$

9. feladat

$$\int_0^{2\pi}\cos x\,dx = \sin x\Big|_0^{2\pi} = \sin 2\pi - \sin 0 = 0$$

10. feladat

$$\int_0^1 x\sinh x\,dx = x\cosh x\Big|_0^1 - \int_0^1 \cosh x\,dx = (x\cosh x - \sinh x)\Big|_0^1$$

$f = x \quad g' = \sinh x$

$f' = 1 \quad g = \cosh x$

$$= \cosh 1 - \sinh 1 = \frac{e+\frac{1}{e}}{2} - \frac{e-\frac{1}{e}}{2} = \frac{1}{e}$$

11. feladat

$$\int_{-3}^{3}\sqrt{9-x^2}\,dx = 3\int_{-3}^{3}\sqrt{1-(x/3)^2}\,dx$$

$\frac{x}{3} = \sin t \quad x = 3\sin t \quad \dot{x} = 3\cos t \quad dx = 3\cos t\,dt$

$$= 3\int_a^b\sqrt{1-\sin^2 t}\,3\cos t\,dt = 9\int_a^b \cos^2 t\,dt = \int_a^b \frac{9}{2}+\frac{9\cos 2t}{2}\,dt =$$

páratlan

$$= \frac{9t}{2}+\frac{9\sin 2t}{4}\Big|_a^b = \frac{9}{2}\arcsin\frac{x}{3} + \frac{9\sin 2\arcsin\frac{x}{3}}{4}\Big|_{-3}^{3} = \frac{9\pi}{2}$$

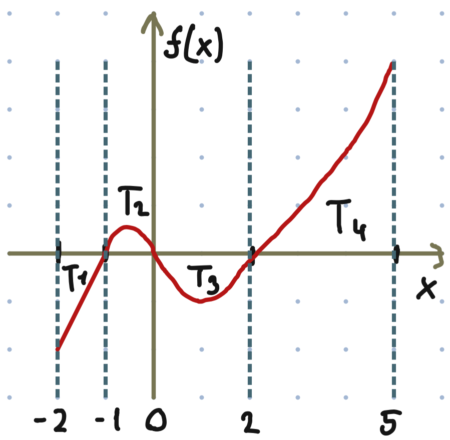
Másik variáció : a és b kiszámítása:

$$\frac{x}{3} = \sin t \to t = \arcsin \frac{x}{3}$$

$$\left.\begin{aligned} a &= \arcsin -1 = -\pi/2 \\ b &= \arcsin +1 = \pi/2 \end{aligned}\right\} \quad \frac{9t}{2} + \frac{9\sin 2t}{4} \Bigg|_{-\pi/2}^{\pi/2} = \frac{9\pi}{2}$$

12. feladat

$$\int_{-2}^{5} |(x+1) \times (x-2)|\,dx = \int_{-2}^{5} |x^3 - x^2 - 2x|\,dx = T_1 + T_2 + T_3 + T_4$$



$$F(x) = \frac{x^4}{4} - \frac{x^3}{3} - x^2$$

$F(-2) = 8/3$

$F(-1) = -5/12$

$F(0) = 0$

$F(2) = -8/3$

$F(5) = 1075/12$

$T_1 = |F(-1) - F(-2)| = F(-2) - F(-1) = 8/3 + 5/12 = 37/12$

$T_2 = F(0) - F(-1) = 0 + 5/12 = 5/12$

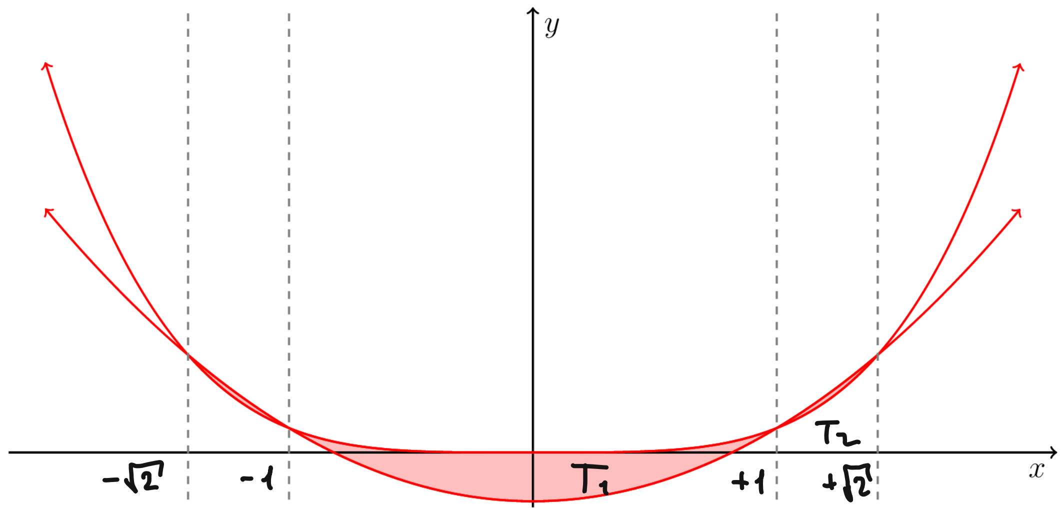
$T_3 = |F(2) - F(0)| = F(0) - F(2) = 0 + 8/3 = 8/3 = 32/12$

$T_4 = F(5) - F(2) = 1075/12 + 32/12 = 1107/12$

$T = \sum T = 1181/12$

13. feladat

$$\begin{cases} f(x) = x^4 \\ g(x) = 3x^2 - 2 \end{cases}$$



Metszéspontok:

$$x^4 = 3x^2 - 2$$

$$x^4 - 3x^2 + 2 = 0$$

$$(x^2 - 2)(x^2 - 1) = 0$$

$$x = \begin{cases} \pm\sqrt{2} \\ \pm 1 \end{cases}$$

páros => elég csak a felét kiszámolni

$$T_1 = \int_0^1 f(x) - g(x)\, dx = \int_0^1 x^4 - 3x^2 + 2\, dx$$

$$= \frac{x^5}{5} - x^3 + 2x \Big|_0^1 = \frac{1}{5} - 1 + 2 = \frac{6}{5}$$

$$T_2 = \int_1^{\sqrt{2}} g(x) - f(x)\, dx = \int_1^{\sqrt{2}} -x^4 + 3x^2 - 2dx$$

$$= \frac{x^5}{5} - x^3 + 2x \Big|_{\sqrt{2}}^{1} = \frac{6}{5} - \frac{4\sqrt{2}}{5} + 2\sqrt{2} - 2\sqrt{2}$$

$$T = 2 \cdot \left(\frac{6}{5} + \frac{6}{5} - \frac{4\sqrt{2}}{5} \right) = \frac{24}{5} - \frac{8\sqrt{2}}{5}$$

14. feladat

$x(t) = a\cos t$

$y(t) = a\sin t$

$t \in [0; 2\pi]$

$\dot{x}(t) = -a\sin t$

$$T = \int_0^{2\pi} |\dot{x} y| \, dt = \int_0^{2\pi} a^2 \sin^2 t \, dt =$$

$$= \int_0^{2\pi} a^2 \, \frac{1 - \cos 2t}{2} \, dt = \frac{a^2 t}{2} \Big|_0^{2\pi} = a^2 \pi$$