Matematika G1 – GY10 Sándor Tibor

1. feladat $f(x) = xe^{-1/x}$

1, $\mathcal{D}_f = \mathbb{R} \setminus \{0\}$

Zérushelyek nincsenek $(0 \notin \mathcal{D}_f)$

Paritás: $f(-x) = -xe^{1/x}$

↳ se nem páros, se nem páratlan

Periodicitás: nincsen

Határértékek:

↳ $\pm\infty$: $\lim\limits_{x \to \pm\infty} xe^{-1/x} = \pm\infty$

$\left(e^{-1/-0} = e^{\infty}\right)$

S

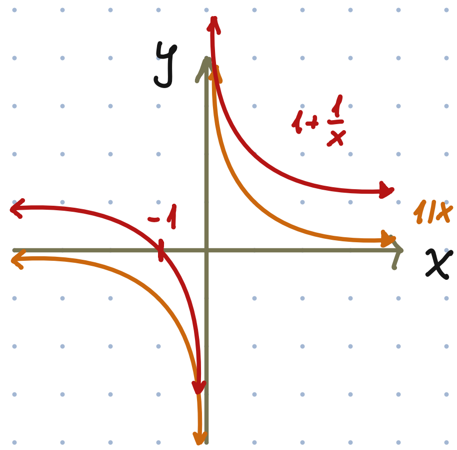
↳ szakadási pontok:

$$\lim_{x \to 0^-} xe^{-1/x} = \lim_{x \to 0^-} \frac{e^{-1/x}}{1/x} = \lim_{x \to 0^-} \frac{e^{-1/x}\left(\cancel{\frac{+1}{x^2}}\right)}{\cancel{-\frac{1}{x^2}}} = \lim_{x \to 0^-} -e^{-1/x} = -\infty$$

$$\lim_{x \to 0^+} xe^{-1/x} = 0$$

↳ $\left(0 \cdot e^{-1/+0} = 0 \cdot e^{-\infty} = 0 \cdot 0 = 0\right)$

2, $f'(x) = e^{-1/x} + xe^{-1/x} \cdot \frac{1}{x^2} = \underbrace{e^{-1/x}}_{>0} \cdot \left(1 + \frac{1}{x}\right)$



$f'(x) < 0$, ha $x \in (-1; 0)$

↳ monoton csökken

$f'(x) > 0$, ha $x \in (-\infty; -1) \cup (0; \infty)$

↳ monoton nő

$f'(x)$ előjelet vált $x = -1$-ben $(\oplus \to \ominus)$

↳ lokális maximum

3, $f''(x) = e^{-1/x} \frac{1}{x^2}\left(1+\frac{1}{x}\right) + e^{-1/x}\left(-\frac{1}{x^2}\right) = \frac{e^{-1/x}}{x^3}$

$f''(x) < 0$, ha $x^3 < 0 \Rightarrow$ konkáv, ha $x \in (-\infty; 0)$

$f''(x) > 0$, ha $x^3 > 0 \Rightarrow$ konvex, ha $x \in (0; +\infty)$

$f''(x)$ sosem 0 ($0 \notin D_f$) $\Rightarrow$ nincs inflexiós pont

4, Aszimptoták

↳ vízszintes: nincsen

↳ függőleges: $x = 0$

↳ ferde: $y = mx + b$

$$\left.\begin{aligned} m_1 &= \lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} e^{-1/x} = 1 \\ m_2 &= \lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} e^{-1/x} = 1 \end{aligned}\right\} \text{ ugyan az}$$

$$b_1 = \lim_{x\to+\infty} f(x) - m_1 x = \lim_{x\to+\infty} xe^{-1/x} - x = \lim_{x\to+\infty} \frac{e^{-1/x} - 1}{1/x} =$$

$$\overset{L'H}{=} \lim_{x\to+\infty} \frac{e^{-1/x}\,\frac{1}{x^2}}{-1/x^2} = \lim_{x\to+\infty} -e^{-1/x} = -e^0 = -1$$

ugyan az

$$b_2 = \lim_{x\to-\infty} f(x) - m_2 x = \lim_{x\to-\infty} \frac{e^{-1/x} - 1}{1/x} = \lim_{x\to-\infty} -e^{-1/x} = -e^0 = -1$$

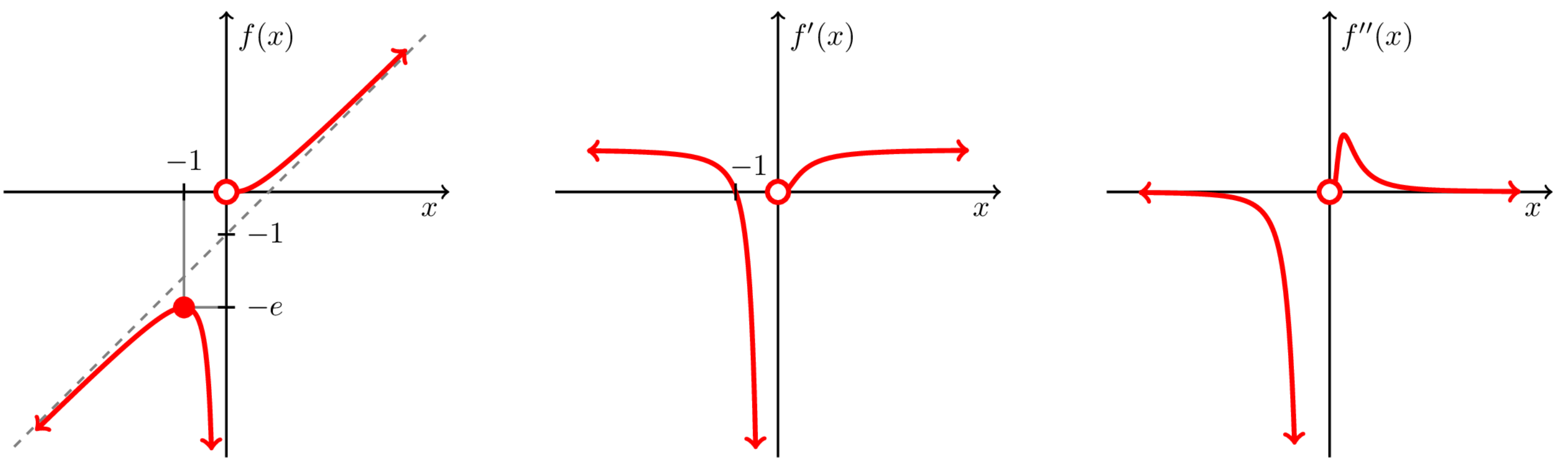
$$y = mx + b = x - 1$$

5, Táblázat készítése

	$(-\infty;-1)$	-1	$(-1;0)$	0	$(0;+\infty)$
$f'(x)$	+	0	−		+
$f''(x)$	−		−		+
$f(x)$	↗	$-e$: max	↘		↗

$\cap$ és ↗ $\quad$ $\cap$ és ↘ $\quad$ $\cup$ és ↗

Grafikon



$$\mathcal{R}_f = \mathbb{R} \setminus (-e;0]$$

2. feladat $f(x) = \sin^2 x - 2\sin x$

$D_f = \mathbb{R}$

ZH: $f(x) = \sin x \underbrace{(\sin x - 2)}_{\in [-3;-1]} = 0$

$\Downarrow$

$\sin x = 0$

$x = k\pi \quad k \in \mathbb{Z}$

Paritás: $f(-x) = \sin^2(-x) - 2\sin(-x) = \sin^2 x + 2\sin x$

$\uparrow$ páros $\quad \uparrow$ páratlan

$\Rightarrow$ se nem páros, se nem páratlan

Periodicitás: $\left.\begin{array}{l} \sin^2 x \quad \pi \text{ szerint} \\ 2\sin x \quad 2\pi \text{ szerint} \end{array}\right\}$ $f(x)$ 2π szerint

Határértékek: $\pm\infty$-ben nem létezik

$f'(x) = 2\sin x\cos x - 2\cos x = 2\cos x(\sin x - 1)$

$f'(x) = 0 \nearrow \cos x = 0 \quad x = \pi/2 + k\pi$
$\quad\quad\quad\;\, \searrow \sin x = 1 \quad x = \pi/2 + 2k\pi$

$f'(x) > 0$, ha $\pi/2 < x + 2k\pi < 3\pi/2$ (mindkét tag $\ominus$)

$f'(x) < 0$, ha $-\pi/2 < x + 2k\pi < \pi/2$ ($\cos x \oplus$, $\sin x - 1 \ominus$)

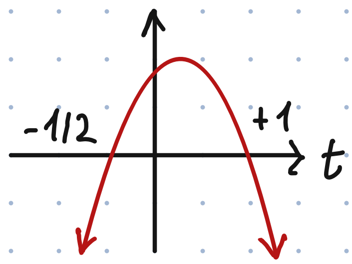
$x = \pi/2 + 2k\pi \to$ minimumok

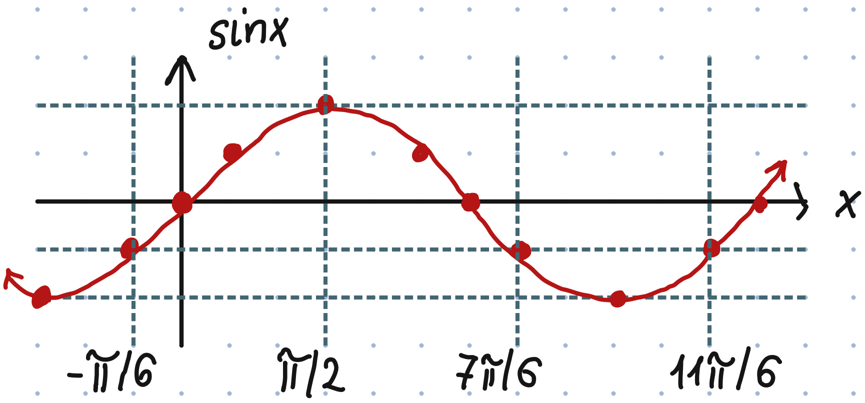
$x = 3\pi/2 + 2k\pi \to$ maximumok

$$f''(x) = -2\sin x(\sin x - 1) + 2\cos^2 x$$
$$= -2\sin^2 x + 2\sin x + 2 - 2\sin^2 x$$
$$= -4\sin^2 x + 2\sin x + 2$$

$$f''(x) = 0 = -4\sin^2 x + 2\sin x + 2 = -2(2\sin^2 x - \sin x + 1)$$
$$0 = 2\sin^2 x - \sin x - 1 = (2\sin x + 1)(\sin x - 1)$$

$$\sin x = \begin{cases} -1/2 \\ +1 \end{cases}$$





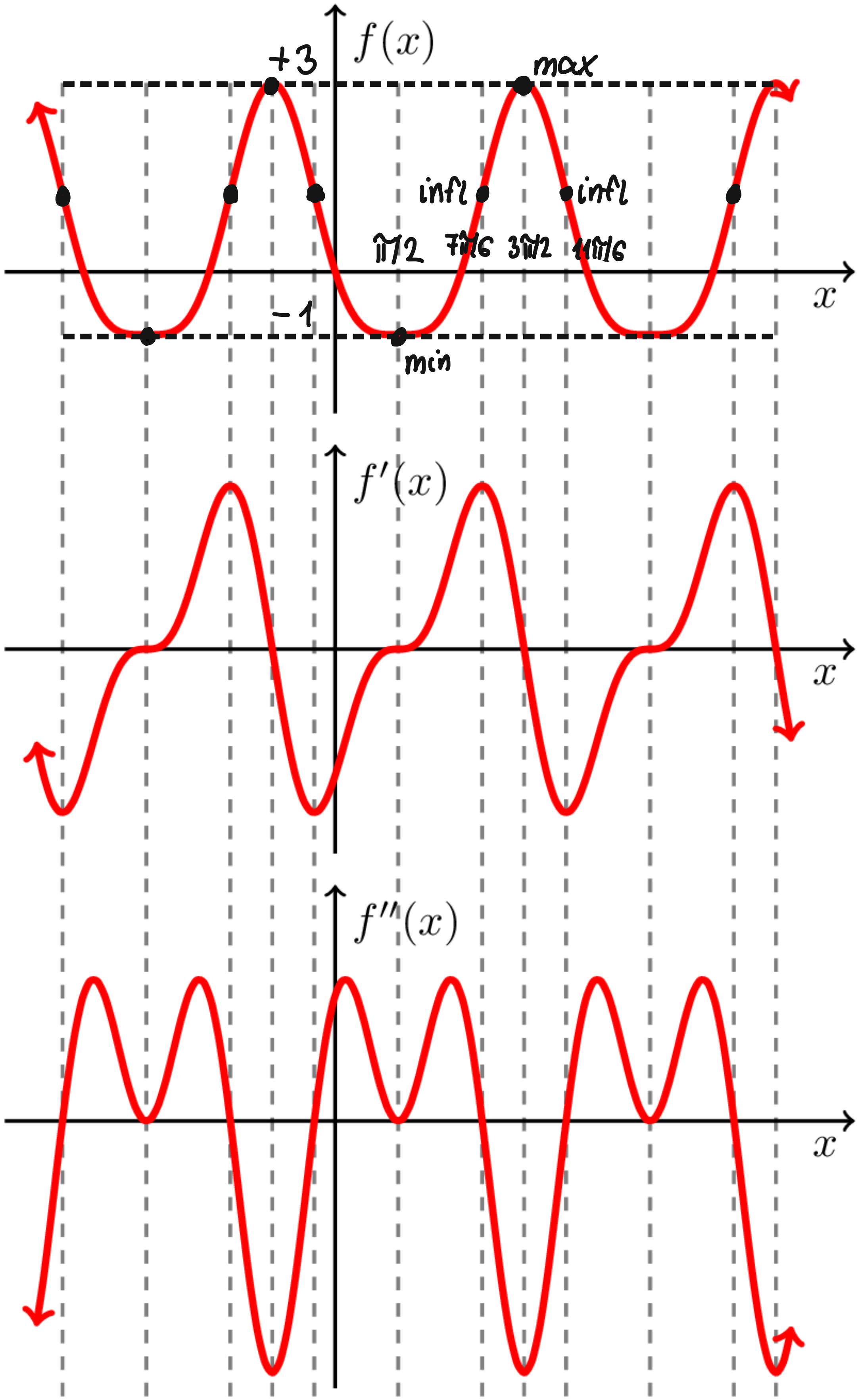
$f''(x) > 0$, ha $-\pi/6 < x + 2k\pi < \pi/2 \quad \vee \quad \pi/2 < x + 2k\pi < 7\pi/6$

$f''(x) < 0$, ha $7\pi/6 < x + 2k\pi < 11\pi/6$

Táblázat

	$-\pi/6<x+2k\pi<\pi/2$	$\pi/2+2k\pi$	$\pi/2<x+2k\pi<7\pi/6$	$7\pi/6+2k\pi$	$7\pi/6<x+2k\pi<3\pi/2$	$3\pi/2+2k\pi$	$3\pi/2<x+2k\pi<11\pi/6$	$11\pi/6+2k\pi$
$f'(x)$	$-$	0	$+$	$+$	$+$	0	$-$	$-$
$f''(x)$	$+$	0	$+$	0	$-$	$-$	$-$	0
$f(x)$	$\nearrow$	-1: min	$\nearrow$	$\frac{3}{4}$: infl	$\searrow$	$+3$: max	$\searrow$	$\frac{3}{4}$: infl

$\mathcal{R}_f = [-1;3]$



3. feladat

a) $$\int x^2(x^2-1)+2\sqrt{x\sqrt{x\sqrt{x}}}\,dx=\int x^4-x^2+2x^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}}\,dx$$

$C\in\mathbb{R}$

$$=\int x^4-x^2+2x^{7/8}=\frac{x^5}{5}-\frac{x^3}{3}+2\,\frac{8}{15}\,x^{15/8}+C=\frac{x^5}{5}-\frac{x^3}{3}+\frac{16x^{15/8}}{15}+C$$

b) $$\int\frac{x^2-4x+7}{x-2}\,dx=\int\frac{(x-2)^2+3}{x-2}\,dx=\int x-2+\frac{3}{x-2}\,dx=$$

$C\in\mathbb{R}$

$$x^2-4x+7=x^2-4x+4+3=(x-2)^2+3$$

$$\frac{x^2}{2}-x+3\ln|x-2|+C$$

c) $$\int\tan^2x\,dx=\int\frac{\sin^2x}{\cos^2x}\,dx=\int\frac{1-\cos^2x}{\cos^2x}\,dx=\int\frac{1}{\cos^2x}-1\,dx$$

$$=\tan x-x+C\qquad C\in\mathbb{R}$$

d) $$\int\frac{\ln 2}{\sqrt{2+2x^2}}+\frac{e^{3x}+1}{e^x+1}\,dx=\int\frac{\ln 2}{\sqrt{2}}\,\frac{1}{\sqrt{1+x^2}}+\frac{\cancel{(e^x+1)}(e^{2x}-e^x+1)}{\cancel{e^x+1}}\,dx \ ^{*}$$

$C\in\mathbb{R}$

$$=\int\frac{\ln 2}{\sqrt{2}}\,\frac{1}{\sqrt{1+x^2}}+e^{2x}-e^x+1\,dx=\frac{\ln 2}{\sqrt{2}}\operatorname{arsinh}x+\frac{e^{2x}}{2}-e^x+x+C$$

* $a^3+b^3=(a+b)(a^2-ab+b^2)$

Helyettesítéses integrálás

$$\int f(g(x)) \cdot g'(x)\, dx = \int f(t)\, dt, \quad \text{ahol} \quad t = g(x) \quad dt = g'(x)dx$$

$$\int \underbrace{f(g(x)) \cdot g'(x)}_{(f \circ g) \cdot g' = (F \circ g)'} dx = \underbrace{F(g(x))}_{(F \circ g)} + C$$

- speciális esetek:

↳ $\int f^{\alpha} \cdot f' = \frac{f^{\alpha+1}}{\alpha+1} + C \qquad (\alpha \neq -1)$

↳ $\int \frac{f'}{f} = \ln|f| + C$

↳ $\int e^{f} \cdot f' = e^{f} + C$

4. feladat

a, $$\int (x^3+2x)\cdot\cos(x^4+4x^2) = \int (x^3+2x)\cdot\cos t\cdot\frac{dt}{4(x^3+2x)} =$$

$t = x^4+4x^2$

$\frac{dt}{dx} = 4x^3+8x^2$

$dx = \frac{dt}{4(x^3+2x^2)}$

$$= \frac{1}{4}\int \cos t\, dt = \frac{1}{4}\sin t + C =$$

$$= \frac{1}{4}\sin(x^4+4x^2) + C \qquad C\in\mathbb{R}$$

b, $$\int \frac{\sqrt[3]{\tan x}}{\cos^2 x}\,dx = \int \frac{\sqrt[3]{t}}{\cos^2 x}\cos^2 x\, dt = \int \sqrt[3]{t}\, dt =$$

$t = \tan x$

$\frac{dt}{dx} = \frac{1}{\cos^2 x}$

$dx = \cos^2 x\, dt$

$$= \frac{3}{4} t^{4/3} + C = \frac{3}{4}\tan^{4/3} x + C$$

$$C\in\mathbb{R}$$

c, $$\int \sin^3(2x+1)\cos(2x+1)\,dx = \int t^3\cos(2x+1)\frac{dt}{2\cos(2x+1)} =$$

$t = \sin(2x+1)$

$\frac{dt}{dx} = 2\cos(2x+1)$

$dx = \frac{dt}{2\cos(2x+1)}$

$$= \frac{1}{2}\int t^3\, dt = \frac{1}{2}\,\frac{t^4}{4} + C =$$

$$= \frac{\sin^4(2x+1)}{8} + C \qquad C\in\mathbb{R}$$

d) $\int \frac{x}{4+x^2}\,dx = \int \frac{x}{t}\,\frac{dt}{2x} = \frac{1}{2}\int \frac{1}{t}\,dt = \frac{\ln|t|}{2} + C$

$t = 4+x^2$

$\frac{dt}{dx} = 2x \qquad dx = \frac{dt}{2x}$

$= \frac{\ln(4+x^2)}{2} + C \qquad C \in \mathbb{R}$

e) $\int \frac{1}{\sin x \cos x}\,dx = \int \frac{1}{\tan x \cos^2 x}\,dx = \int \frac{1}{t}\,dt = \ln|t| + C$

$= \ln|\tan x| + C$

$t = \tan x \qquad dx = \cos^2 x\,dt$

$C \in \mathbb{R}$

f) $\int \frac{1}{\tan x}\,dx = \int \frac{\cos x}{\sin x}\,dx = \int \frac{1}{t}\,dt = \ln|t| + C = \ln|\sin x| + C$

$C \in \mathbb{R}$

$t = \sin x \qquad dx = \frac{dt}{\cos x}$

g) $\int \frac{1}{x \ln x}\,dx = \int \frac{1}{t}\,dt = \ln|t| + C = \ln|\ln x| + C \qquad C \in \mathbb{R}$

$t = \ln x \qquad dx = x\,dt$

h) $\int \frac{2x+1}{x^2+2} = \int \frac{2x}{x^2+2} + \frac{1}{x^2+2}\,dx = \ln|x^2+2| + \frac{1}{\sqrt{2}}\arctan\frac{x}{\sqrt{2}} + C$

$C \in \mathbb{R}$

$\int \frac{1}{x^2+2}\,dx = \int \frac{1/2}{(x/\sqrt{2})^2 + 1}\,dx = \frac{\sqrt{2}}{2}\int \frac{1}{t^2+1}\,dt = \arctan t + C$

$t = x/\sqrt{2}$

$dx = \sqrt{2}\,dt$

$= \arctan\frac{x}{\sqrt{2}} + C$

Érdekesség:

$$\int \sin x \cos x \, dx$$

a, $\frac{1}{2}\int \sin 2x \, dx = \frac{1}{4}\int \sin t \, dt = \frac{-\cos 2x}{4} + C_1$

$t = 2x$

$\frac{dt}{dx} = 2 \quad dx = \frac{dt}{2}$

b, $\int \sin x \cos x \, dx = \int t \, dt = \frac{t^2}{2} + C_1 = \frac{\sin^2 x}{2} + C_1$

$t = \sin x$

$\frac{dt}{dx} = \cos x \quad dx = \frac{dt}{\cos x}$

c, $\int \sin x \cos x \, dx = \int -t \, dt = \frac{-\cos^2 x}{2} + C_1$

$t = \cos x$

$\frac{dt}{dx} = -\sin x \quad dx = \frac{-dt}{\sin x}$