Matematika G1 - GY8 Sándor Tibor

1. feladat

$$x^4y+5y^2x-4=0$$

Parciális deriválással:

$$\left.\begin{aligned}\frac{\partial F}{\partial x}&=4x^3y+5y^2\\ \frac{\partial F}{\partial y}&=x^4+10yx\end{aligned}\right\}\Rightarrow \begin{aligned}\frac{dy}{dx}&=-\frac{F_x'}{F_y'}=-\frac{4x^3y+5y^2}{x^4+10yx}\\ \frac{dx}{dy}&=-\frac{F_y'}{F_x'}=-\frac{x^4+10yx}{4x^3y+5y^2}\end{aligned}$$

Implicit deriválással:

$$\xrightarrow{dy/dx}\quad 4x^3y+x^4y'+5y^2+10yy'x=0$$

$$y'(x^4+10yx)=-(4x^3y+5y^2)$$

$$y'=\frac{dy}{dx}=-\frac{4x^3y+5y^2}{x^4+10yx}$$

$$\xrightarrow{dx/dy}\quad 4x^3x'y+x^4+5y^2x'+10yx=0$$

$$x'(4x^3y+5y^2)=-(x^4+10yx)$$

$$x'=\frac{dx}{dy}=-\frac{x^4+10yx}{4x^3y+5y^2}$$

2. feladat

$\ln y + xy = 1$ $\quad P_0(1;1)$

Ellenőrizzük, hogy P_0 rajta van-e a görbén!

$\ln 1 + 1\cdot 1 = 0 + 1 = 1$ ✓

Első derivált: (dy/dx)

$$\frac{1}{y}y' + y + xy' = 0$$

$$y'\left(\frac{1}{y} + x\right) = -y \to y' = \frac{-y}{1/y + x} = \frac{-y^2}{1+xy}$$

$$y'|_{P_0} = \frac{-1^2}{1+1\cdot 1} = -\frac{1}{2}$$

Második derivált: (d^2y/dx^2)

$$y'' = \frac{\frac{d}{dx}(-y^2)(1+xy) - (-y^2)\frac{d}{dx}(1+xy)}{(1+xy)^2}$$

$$\frac{d}{dx}(-y^2) = -2yy' \qquad \frac{d}{dx}(1+xy) = y + xy'$$

$$y'' = \frac{-2y\left(\frac{-y^2}{1+xy}\right)(1+xy) + y^2\left(y - \frac{xy^2}{1+xy}\right)}{(1+xy)^2} = \frac{2y^3 + y^3 - \frac{xy^4}{1+xy}}{(1+xy)^2}$$

$$y'' = \frac{3y^3(1+xy) - xy^4}{(1+xy)^3} = \frac{3y^3 + 3xy^4 - xy^4}{(1+xy)^3} = \frac{3y^3 + 2xy^4}{(1+xy)^3}$$

3. feladat $x^2+y^2=25$ $m=3/4$

$\xrightarrow{dy/dx}$ $2x+2yy'=0 \rightarrow y'=-\frac{x}{y}$

$y'=3/4 \rightarrow -x/y=3/4$ $y=\frac{-4x}{3}$

$x^2+\frac{16x^2}{9}=25 \rightarrow \frac{25x^2}{9}=25 \rightarrow x^2=9 \rightarrow x=\pm 3$

$x_1=3 \rightarrow y_1=-4$

$x_2=-3 \rightarrow y_2=4$

4. feladat $f(x)=5x^3+x-7$ $x_0=1$

$f'(x)=15x^2+1>0 \Rightarrow$ monoton nő $\Rightarrow \exists! f^{-1}$

$x_0=1 \rightarrow y_0=f(x_0)=5+1-7=-1$

$f'(x_0)=15+1=16$

$f^{-1'}(y_0)=\frac{1}{16}$

$f^{-1'}=\frac{1}{f'}=\frac{1}{15x^2+1}$

5. **feladat** $x(t) = e^t \cos t$ $y(t) = e^t \sin t$

$$\left.\begin{aligned} \dot x &= e^t(\cos t - \sin t) \\ \dot y &= e^t(\sin t + \cos t) \end{aligned}\right\} \quad y' = \frac{\dot y}{\dot x} = \frac{\sin t + \cos t}{\cos t - \sin t}$$

$$y'\Big|_{t=\pi/6} = \frac{1/2 + \sqrt{3}/2}{\sqrt{3}/2 - 1/2} = \frac{1+\sqrt{3}}{\sqrt{3}-1}$$

$$\ddot x = e^t(\cos t - \sin t - \sin t - \cos t) = -2e^t \sin t$$

$$\ddot y = e^t(\sin t + \cos t + \cos t - \sin t) = 2e^t \cos t$$

$$y'' = \frac{\ddot y \dot x - \dot y \ddot x}{\dot x^3} = \frac{e^{2t}(+2\cos t \cdot (\cos t - \sin t) + 2\sin t(\sin t + \cos t))}{e^{3t}(\cos t - \sin t)^3}$$

$$= \frac{2(\cos^2 t - \cos t \sin t + \sin^2 t + \sin t \cos t)}{e^t(\cos t - \sin t)^3} = \frac{2}{e^t(\cos t - \sin t)^3}$$

6. **feladat** $x(t) = 5\cos t$ $y(t) = 5\sin t$ $m = 3/4$

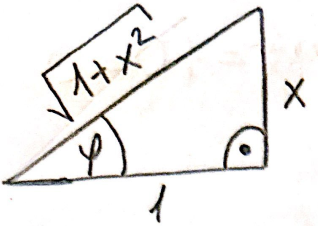
$$\left.\begin{aligned} \dot x &= -5\sin t \\ \dot y &= 5\cos t \end{aligned}\right\} \quad y' = \frac{\dot y}{x} = -\cot t$$

$y' = 3/4 \to \cot t = -3/4 \to \tan t = -4/3$

$$\left.\begin{aligned} x_1 &= 5\cos\arctan(-4/3) = 3 \\ y_1 &= 5\sin\arctan(-4/3) = -4 \end{aligned}\right\}$$

$$\left.\begin{aligned} x_2 &= 5\cos(\arctan(-4/3) + \pi) = -3 \\ y_2 &= 5\sin(\arctan(-4/3) + \pi) = 4 \end{aligned}\right\}$$

kitekintő:



$$\tan\varphi=\frac{x}{1}=x$$

$$\varphi=\arctan x$$

$$\sin\arctan x=\sin\varphi=\frac{x}{\sqrt{1+x^2}}$$

$$\cos\arctan x=\cos\varphi=\frac{1}{\sqrt{1+x^2}}$$

7. feladat

a, $\lim\limits_{x\to3}\frac{3^x-x^3}{x-3}\overset{L'H}{=}\lim\limits_{x\to3}\frac{3^x\ln3-3x^2}{1}=27\ln3-27$

b, $\lim\limits_{x\to3^+}\frac{\ln(x-3)}{\ln(e^x-e^3)}\overset{L'H}{=}\lim\limits_{x\to3^+}\frac{\frac{1}{x-3}}{\frac{1}{e^x-e^3}e^x}=\lim\limits_{x\to3^+}\frac{e^x-e^3}{e^x(x-3)}=$

$=\lim\limits_{x\to3^+}\frac{e^x}{e^x(x-3)+e^x}=\lim\limits_{x\to3^+}\frac{1}{x-3+1}=\frac{1}{3-3+1}=1$

c, $\lim\limits_{x\to0}x\ln x=\lim\limits_{x\to0}\frac{\ln x}{\frac{1}{x}}\overset{L'H}{=}\lim\limits_{x\to0}\frac{1/x}{-1/x^2}=\lim\limits_{x\to0}-x=0$

d, $\lim\limits_{x\to0}x^x=\lim\limits_{x\to0}e^{x\ln x}=e^0=1$

e, $\lim\limits_{x\to0}\left(\frac{\cos x}{x}-\frac{1}{\sin x}\right)=\lim\limits_{x\to0}\frac{\cos x\sin x-x}{x\sin x}=$

$=\lim\limits_{x\to0}\frac{-\sin^2x+\cos^2x-1}{\sin x+x\cos x}=\lim\limits_{x\to0}\frac{\cos2x-1}{\sin x+x\cos x}\overset{L'H}{=}$

$=\lim\limits_{x\to0}\frac{-2\sin2x}{2\cos x-x\sin x}=\frac{0}{2+0}=0$

8. feladat

$$\lim_{x\to 0}\frac{x^2\sin(1/x)}{\sin x}=\lim_{x\to 0}\overset{\nearrow 1}{\left(\frac{x}{\sin x}\right)}\overset{\nearrow 0}{\left(x\sin\frac{1}{x}\right)}=1\cdot 0=0$$

L'Hôpital szabállyal:

$$\lim_{x\to 0}\frac{2x\sin(1/x)-\cos\frac{1}{x}}{\cos x}=\lim_{x\to 0}-\cos\frac{1}{x}\ ↯$$

9. feladat

$f(x)=3x^5+15x-2$

$$\left.\begin{array}{l}x=0\to f(0)=-2\\ x=1\to f(1)=16\end{array}\right\}\Rightarrow$$

$[0;1]$-en előjelet vált

$\Rightarrow$ Bolzano

$\Rightarrow$ Van legalább 1 gyök

TFH: 2 gyök van.

f folytonos $\Rightarrow$ Rolle tétel: $\exists\xi: f'(\xi)=0$

$f'(x)=15x^4+15>0$ ↯ $\Rightarrow$ csak 1 gyök van.

10. feladat

$$f(x) = \frac{x^3}{x^2 - x - 2}$$

$$f(x) = \frac{x^3}{(x-2)(x+1)} \quad : \mathcal{D}_f = \mathbb{R} \setminus \{-1;2\}$$

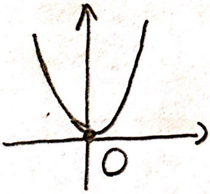
$$f'(x) = \frac{3x^2(x-2)(x+1) - x^3(2x-1)}{(x-2)^2(x+1)^2}$$

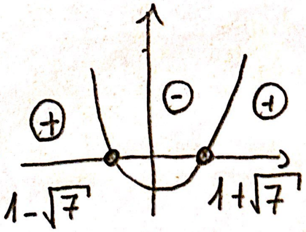
$$f'(x) = \frac{x^2(3x^2-3x-6-2x^2+x)}{(x-2)^2(x+1)^2} = \frac{x^2(x^2-2x-6)}{(x-2)^2(x+1)^2}$$

$$f'(x) = \frac{x^2(x^2-2x-6)}{(x-2)^2(x+1)^2} \qquad f'(x) = 0$$

$$0 = x^2(x^2-2x-6)$$

↙ ↘





↑ ↓ ↘

mindig ⊕

nincs sz.é.

lok max

lok min