Matematika G1 - GY7 Sándor Tibor

1. feladat $f(x) = x^n$

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x \to x_0} \frac{x^n - x_0^n}{x-x_0}$$

$$x^n - x_0^n = (x-x_0)(x^{n-1} + x^{n-2}x_0 + x^{n-3}x_0^2 + \dots + x_0^{n-1})$$

$$f'(x_0) = \lim_{x \to x_0} \frac{(x-x_0)(\dots)}{(x-x_0)} = \lim_{x \to x_0} (x^{n-1} + x^{n-2}x_0 + \dots + x_0^{n-1})$$

$$= \underbrace{x_0^{n-1} + x_0^{n-1} + \dots + x_0^{n-1}}_{n \text{ darab}} = n x_0^{n-1}$$

Másik mo.: $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$

$$(x+h)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} h^k = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \dots + \binom{n}{n}h^n$$

$$f'(x) = \lim_{h \to 0} \frac{\cancel{x^n} + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \dots + h^n - \cancel{x^n}}{h}$$

$$= \lim_{h \to 0} nx^{n-1} + \binom{n}{2}x^{n-2}h + \dots + h^{n-1}$$

$$= nx^{n-1} + 0 + \dots + 0 = nx^{n-1}$$

2. feladat

a, $f(x) = \begin{cases} \sin^2 x, & \text{ha } x \leq 0 \\ x^2, & \text{ha } x > 0 \end{cases}$

differenciálhányados létezésének szükséges feltétele: folytonosság

$$\left.\begin{aligned} \lim_{x \to 0^-} f(x) &= \lim_{x \to 0^-} \sin^2 x = 0 \\ \lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} x^2 = 0 \end{aligned}\right\} \text{folytonos}$$

deriválás def szerint:

$$\lim_{x \to 0^-} \frac{\sin^2 x - \sin^2(0)}{x - 0} = \lim_{x \to 0^-} \frac{\sin x}{x} \sin x = 1 \cdot 0 = 0$$

$$\lim_{x \to 0^+} \frac{x^2 - 0^2}{x - 0} = \lim_{x \to 0^+} x = 0$$

$=$

$\Downarrow$

$f'(0) = 0$

$$f'(x) = \begin{cases} \sin 2x & \text{ha } x < 0 \\ 0 & \text{ha } x = 0 \\ 2x & \text{ha } x > 0 \end{cases}$$

b, $f(x) = \begin{cases} \arctan x, & \text{ha } x > 0 \\ 0, & \text{ha } x = 0 \\ x^3 + x + 1, & \text{ha } x < 0 \end{cases}$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \arctan x = 0$$

$\neq \Rightarrow \nexists f'(0)$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^3 + x + 1 = 1$$

3. feladat

a, $|x-1|$

b, nincs ilyen

c, $0; x; x^2$

d, $f(x) = \begin{cases} 0, & \text{ha } x<0 \\ x^2, & \text{ha } x \geq 0 \end{cases}$

4. feladat

0-ban: $f'(0) = \lim\limits_{x \to 0} \dfrac{x^2\left|\sin \frac{1}{x}\right| - 0}{x-0} = \lim\limits_{x\to 0} x\left|\sin \frac{1}{x}\right| - 0$

↳ tehát a 0-ban diffható

$\left|\sin \frac{1}{x}\right| = 0$ problémás

↳ $x = \frac{1}{k\pi}$ $k \in \mathbb{Z}$ helyeken nem diffható

5. feladat

a, $f(x) = \underbrace{(6x^7+7x^4+2x^2)^5}_{a(b+c+d)} + \underbrace{\sin^2 x}_{e(f)} + \underbrace{\cos^2 x}_{h(i)}$

$f'(x) = \underbrace{5(6x^7+7x^4+2x^2)^4}_{a'(b+c+d)} \cdot \underbrace{(42x^6+28x^3+4x)}_{b+c+d} + \underbrace{2\sin x}_{e'(f)}\underbrace{\cos x}_{f'} \overbrace{+2\cos x}^{h'(i)}\overbrace{(-\sin x)}^{i'}$

b) $g(x) = \underbrace{\ln x \cdot e^{x}}_{a \cdot b} + \underbrace{x^{2} \cot x}_{c \cdot d} + \underbrace{x^{-1/3}}_{e}$

$$g'(x) = \underbrace{\frac{1}{x} e^{x}}_{a'b} + \underbrace{\ln x \, e^{x}}_{a b'} + \underbrace{2x \cot x}_{c'd} + \underbrace{x^{2} \frac{-1}{\sin^{2} x}}_{c d'} + \underbrace{\left(-\frac{1}{3}\right) x^{-4/3}}_{e'}$$

c) $h(x) = \dfrac{\overbrace{(3x+x^{2})}^{a+b} \overbrace{\sinh x}^{c} \overbrace{\arctan x}^{d}}{\underbrace{(1+\cos x)}_{e} \underbrace{\operatorname{artanh} \pi}_{K}} \sim \dfrac{(a+b)\, c\, d}{e \cdot K}$

$$h'(x) = \frac{1}{\operatorname{artanh} \pi} \left[\frac{\{(3+2x) \sinh x \arctan x + (3x+x^{2}) \cosh x \arctan x +}{(1+\cos x)^{2}} \right.$$

$$\left. + \frac{(3x+x^{2}) \sinh x \frac{1}{1+x^{2}} \} (1+\cos x) - \{(3x+x^{2}) \sinh x \arctan x\} (-\sin x)}{1+\cos} \right]$$

$$\left(\frac{(a+b)cd}{e \cdot K} \right)' = \frac{1}{K} \cdot \frac{\{(a+b)'cd + (a+b)c'd + (a+b)cd'\} e - \{(a+b)cd\} e'}{e^{2}}$$

d) $i(x) = \sqrt[3]{\ln \cos^{2} x^{4}}$ $\qquad a(b(c(d(e))))$

$a(x) = \sqrt[3]{x}$ $\quad b(x) = \ln x$ $\quad c(x) = x^{2}$ $\quad d(x) = \cos x$ $\quad e(x) = x^{4}$

$$i'(x) = \underbrace{\frac{1}{3} (\ln \cos^{2} x^{4})^{-2/3}}_{a'(b(c(d(e))))} \cdot \underbrace{\frac{1}{\cos^{2} x^{4}}}_{b'(c(d(e)))} \cdot \underbrace{2 \cos x^{4}}_{c'(d(e))} \cdot \underbrace{(-\sin x^{4})}_{d'(e)} \cdot \underbrace{4x^{3}}_{e'}$$

e, $j(x) = \ln \arcsin \sqrt{\frac{x^2+3}{e^{2x}}}$ $\quad a(b(c(\frac{d}{e})))$

$$j'(x) = \underbrace{\frac{1}{\arcsin\sqrt{\frac{x^2+3}{e^{2x}}}}}_{a'(b(c(\frac{d}{e})))} \cdot \underbrace{\frac{1}{\sqrt{1-\left(\frac{x^2+3}{e^{2x}}\right)}}}_{b'(c(\frac{d}{e}))} \cdot \underbrace{\frac{1}{2}\left(\frac{x^2+3}{e^{2x}}\right)^{-\frac{1}{2}}}_{c'(\frac{d}{e})} \cdot \underbrace{\left(\frac{2xe^{2x}-(x^2+3)2e^{2x}}{e^{4x}}\right)}_{\frac{d'e-de'}{e^2}}$$

f, $k(x) = x^x$

$$x^x = e^{\ln x^x} = e^{x\ln x}$$

$$(x^x)' = (e^{x\ln x})' = e^{x\ln x}\left(1\cdot\ln x + x\cdot\frac{1}{x}\right) = x^x(\ln x + 1)$$

g, $l(x) = (x^2+1)^{\sin x}$

$$(x^2+1)^{\sin x} = e^{\ln(x^2+1)^{\sin x}} = e^{\sin x \ln(x^2+1)}$$

$$l'(x) = \left(e^{\sin x\cdot\ln(x^2+1)}\right)' = e^{\sin x\ln(x^2+1)}\left[\cos x \ln(x^2+1) + \sin x \frac{1}{x^2+1}2x\right]$$

$$l'(x) = (x^2+1)^{\sin x}\left[\cos x\ln(x^2+1) + \frac{2x\sin x}{x^2+1}\right]$$

6. feladat

a, $f(x) = x^m \qquad (x^m)^{(n)} = ?$

$f'(x) = mx^{m-1}$

$f''(x) = m(m-1)x^{m-2}$

$f^{(n)}(x) = m(m-1)\dots(m-n+1)x^{m-n}$

$\Uparrow$ $m>n$ esetén igaz

$m=n$ esetén: $f^{(n)}(x) = 1$

$m<n$ esetén: $f^{(n)}(x) = 0$

b, $f(x) = \sin x \qquad (\sin x)^{(n)} = ?$

$$\left.\begin{aligned}(\sin x)' &= \cos x = \sin\left(x+\frac{\pi}{2}\right)\\ (\sin x)'' &= -\sin x = \sin(x+\pi)\\ (\sin x)''' &= -\cos x = \sin\left(x+\frac{3\pi}{2}\right)\\ (\sin x)'''' &= \sin x = \sin\left(x+\frac{4\pi}{2}\right)\end{aligned}\right\}$$

Sejtés: $(\sin x)^{(n)} = \sin\left(x+\frac{n\pi}{2}\right)$

Bizonyítás: teljes indukció

- első pár eset ✓
- TFH: $(\sin x)^{(k)} = \sin\left(x+\frac{k\pi}{2}\right)$
- $(k+1)$-re: $(\sin x)^{(k+1)} = \left(\sin\left(x+\frac{k\pi}{2}\right)\right)' = \cos\left(x+\frac{k\pi}{2}\right) =$

$= \sin\left(\frac{\pi}{2}-x-\frac{k\pi}{2}\right) = \sin\left(\pi-\frac{\pi}{2}+x+\frac{k\pi}{2}\right) = \sin\left(x+\frac{(k+1)\pi}{2}\right)$ ✓

7. feladat

$$f(x) = 2x^3 + 3\sqrt{x} - 3/2x^2 \qquad x_0 = 1$$

$$f(x_0) = 2+3-3/2 = 7/2$$

$$f'(x) = 6x^2 + \frac{3}{2\sqrt{x}} - \frac{3}{2}(-2)x^{-3} = 6x^2 + \frac{3}{2\sqrt{x}} + 3x^{-3}$$

$$f'(x_0) = 6 + 3/2 + 3 = 21/2$$

$\rightarrow$ érintő: $y = f'(x_0)(x-x_0) + f(x_0)$

$$y = \frac{21}{2}(x-1) + \frac{7}{2} = \frac{21x}{2} - 7$$

$\rightarrow$ merőleges: $y = \frac{-1}{f'(x_0)}(x-x_0) + f(x_0)$

$$y = \frac{-2}{21}(x-1) + \frac{7}{2} = \frac{-2x}{21} + \frac{151}{42}$$

8. feladat

$$x^2 + y^2 = 25 \rightarrow y = f(x) = \pm\sqrt{25-x^2}$$

$$e: 3x - 4y + 7 = 0 \rightarrow y = \frac{3}{4}x + \frac{7}{4}$$

$\rightarrow$ meredekség

$$f'(x) = \pm \frac{1/2}{\sqrt{25-x^2}}(-2x) = \mp \frac{x}{\sqrt{25-x^2}} = \frac{3}{4} \Rightarrow x = \pm 3, \; y = \pm 4$$

Helyes pontok: $(-3;4)$ és $(3;-4)$