Matematika G1 GY6 Sándor Tibor

1. feladat

$$\lim_{x\to 2} \frac{3x+1}{5x+4} = \frac{1}{2}$$

⇝ az f függvény értelmezési tartománya:
$\mathcal{D}_f = \mathbb{R} \setminus \{-4/5\} \Rightarrow$ f értelmezve van a 2-ben, és annak környezetében

⇝ Heine-def: $!\, x_n$ tetszőleges sorozat, melyre $\lim_{n\to\infty} x_n = 2$

$$\lim_{n\to\infty} \frac{3x_n+1}{5x_n+4} = \frac{3\lim_{n\to\infty} x_n + 1}{5\lim_{n\to\infty} x_n + 4} = \frac{3\cdot 2+1}{5\cdot 2+4} = \frac{7}{14} = \frac{1}{2}$$

⇝ Cauchy-def: $|f(x)-A| < \varepsilon$, ha $0<|x-a|<\delta(\varepsilon)$

$$\left|\frac{3x+1}{5x+4} - \frac{1}{2}\right| = \left|\frac{2(3x+1)-(5x+4)}{2(5x+4)}\right| = \left|\frac{6x+2-5x-4}{10x+8}\right| = \left|\frac{x-2}{10x+8}\right| < \varepsilon$$

x az, $a=2$' δ sugarú környezetében van

↳ $10x+8>0 \quad x-2 \gtrless 0 \to x \gtrless 2$

$x>2$ esete:

$x-2 < \varepsilon(10x+8)$

$x(1-10\varepsilon) < 8\varepsilon+2$

$x < \dfrac{2+8\varepsilon}{1-10\varepsilon}$

$x<2$ esete:

$-(x-2) < \varepsilon(10x+8)$

$-x(1+10\varepsilon) < 8\varepsilon-2$

$x > \dfrac{2-8\varepsilon}{1+10\varepsilon}$

Összegezve:

$2-\delta < x < 2+\delta$

$2-\delta = \dfrac{2-8\varepsilon}{1+10\varepsilon} \qquad 2+\delta = \dfrac{2+8\varepsilon}{1-10\varepsilon}$

$\delta_1 = 2 - \dfrac{2-8\varepsilon}{1+10\varepsilon} = \dfrac{28\varepsilon}{1+10\varepsilon}$

$\delta_2 = \dfrac{2+8\varepsilon}{1-10\varepsilon} - 2 = \dfrac{28\varepsilon}{1-10\varepsilon}$

$!\exists\, \delta(\varepsilon) \Rightarrow \exists$ a határérték

2. feladat

a, $\lim_{x\to-\infty}\left(\frac{x^2}{2x+1}+\frac{x^3+4x^2-2}{1-2x^2}\right)$

$$\lim_{x\to-\infty}\frac{x^2(1-2x^2)+(x^3+4x^2-2)(2x+1)}{(2x+1)(1-2x^2)}$$

$$\lim_{x\to-\infty}\frac{x^2-2x^4+2x^4+8x^3-4x+x^3+4x^2-2}{2x+1-4x^3-2x^2}$$

$$\lim_{x\to-\infty}\frac{9x^3+5x^2-4x-2}{-4x^3-2x^2+2x+1}=-\frac{9}{4}$$

b, $\lim_{x\to 0}\frac{3}{2^{\sqrt{2}/x}+1}$ $\to$ nincs a 0-ban értelmezve Vizsgáljuk a bal és jobb oldali határértékeket!

$$\lim_{x\to 0^-}\frac{3}{2^{\sqrt{2}/x}+1}=3\quad\left(\sim\frac{3}{2^{-\infty}+1}\right)$$
$$\neq$$
$$\lim_{x\to 0^+}\frac{3}{2^{\sqrt{2}/x}+1}=0\quad\left(\sim\frac{3}{2^{\infty}+1}\right)$$

A két határérték nem egyezik meg $\to \nexists \lim_{x\to 0} f(x)$

c, $\lim_{x\to-1}\frac{x+1}{\sqrt{6x^2+3}+3x}$ Bővítsünk $(\sqrt{6x^2+3}-3x)$-szel!

$$\lim_{x\to-1}\frac{(x+1)(\sqrt{6x^2+3}-3x)}{6x^2+3-9x^2}=\lim_{x\to-1}\frac{(x+1)(\sqrt{6x^2+3}-3x)}{-3(x^2-1)}=$$

$$=\lim_{x\to-1}\frac{(x+1)(\sqrt{6x^2+3}-3x)}{-3(x+1)(x-1)}=\lim_{x\to-1}\frac{\sqrt{6x^2+3}-3x}{-3(x-1)}=\frac{\sqrt{6+3}+3}{-3(-2)}=1$$

d, $\lim\limits_{x\to 1}\left(\frac{1+x}{2+x}\right)^{\frac{1-\sqrt{x}}{1-x}}$

$$\lim_{x\to 1}\left(\frac{1+x}{2+x}\right)^{\frac{1-\sqrt{x}}{(1+\sqrt{x})(1-\sqrt{x})}} = \left(\frac{1+1}{1+2}\right)^{\frac{1}{1+\sqrt{1}}} = \left(\frac{2}{3}\right)^{\frac{1}{2}} = \sqrt{\frac{2}{3}} = \frac{\sqrt{6}}{3}$$

e, $\lim\limits_{x\to \pi/6}\frac{2\sin^2 x+\sin x-1}{2\sin^2 x-3\sin x+1}$

$\sin x = y \qquad \sin \pi/6 = 1/2$

$$\lim_{y\to 1/2}\frac{2y^2+y-1}{2y^2-3y+1} = \lim_{y\to 1/2}\frac{(2y-1)(y+1)}{(2y-1)(y-1)} = \frac{3/2}{-1/2} = -3$$

$$2y^2+y-1 = (2y+\Delta)(y+\square) = (2y-1)(y+1)$$

$$\left.\begin{aligned} 2\square+\Delta &= 1 \\ \square\Delta &= -1 \end{aligned}\right\} \begin{aligned} \square &= 1 \\ \Delta &= -1 \end{aligned}$$

$$2y^2-3y+1 = (2y+\Delta)(y+\square) = (2y-1)(y-1)$$

$$\left.\begin{aligned} 2\square+\Delta &= -3 \\ \square\Delta &= 1 \end{aligned}\right\} \begin{aligned} \square &= -1 \\ \Delta &= -1 \end{aligned}$$

f, $\lim\limits_{x\to \pi/2}\frac{\cos x-\sin x+1}{\cos x+\sin x-1}$

$\cos^2\frac{x}{2} = \frac{1+\cos x}{2} \to 2\cos^2\frac{x}{2} = 1+\cos x$

$\sin^2\frac{x}{2} = \frac{1-\cos x}{2} \to -2\sin^2\frac{x}{2} = 1-\cos x$

$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$

$$\lim_{x\to \pi/2}\frac{2\cos^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}{-2\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$\lim_{x\to \pi/2}\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\cdot\frac{(\cos\frac{x}{2}-\sin\frac{x}{2})}{(\cos\frac{x}{2}-\sin\frac{x}{2})} = \lim_{x\to \pi/2}\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}} = \frac{1/\sqrt{2}}{1/\sqrt{2}} = 1$$

g, $\lim\limits_{x \to 0} \frac{\sin 5x}{x}$

Tudjuk, hogy $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$

$\Downarrow$

$\lim\limits_{x \to 0} \frac{\sin 5x}{x} = 5 \cdot \lim\limits_{x \to 0} \frac{\sin 5x}{5x} = 5 \cdot 1 = 5$

h, $\lim\limits_{x \to 0} \frac{\tan x - \sin x}{x^3}$

$\lim\limits_{x \to 0} \frac{\frac{\sin x}{\cos x} - \sin x}{x^3} = \lim\limits_{x \to 0} \frac{\sin x (1 - \cos x)}{x^3 \cos x} =$

$= \lim\limits_{x \to 0} \frac{\sin x}{x} \cdot \frac{2\sin^2 \frac{x}{2}}{\left(\frac{x}{2}\right)^2} \cdot \frac{1}{4} \cdot \frac{1}{\cos x} = 1 \cdot 2 \cdot \frac{1}{4} \cdot 1 = \frac{1}{2}$

i, $\lim\limits_{x \to 0} \frac{\sin 8x}{\tan 5x}$

$\lim\limits_{x \to 0} \frac{\sin 8x}{8x} \cdot \frac{5x}{\sin 5x} \cdot \frac{8x}{5x} \cdot \cos 5x = \frac{8}{5}$

j, $\lim\limits_{x \to \pi/2} (\pi/2 - x) \tan x$ Legyen $t = \pi/2 - x$ $\begin{matrix} x \to \pi/2 \\ t \to 0 \end{matrix}$

$\lim\limits_{t \to 0} t \cdot \frac{\sin(\pi/2 - t)}{\cos(\pi/2 - t)} = \lim\limits_{t \to 0} \frac{t \cos t}{\sin t} = 1$

k, $\lim\limits_{x \to 0} \frac{1 - \cos \sin x}{\sin^2 x}$ $t = \sin x$ $\begin{matrix} x \to 0 \\ t \to 0 \end{matrix}$

$\lim\limits_{t \to 0} \frac{1 - \cos t}{t^2} = \lim\limits_{t \to 0} \frac{2\sin^2 \frac{t}{2}}{(t/2)^2} \cdot \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$

3. feladat

$$f(x) = \frac{x^2+3x-10}{x^2-3x+2}$$

$$x^2+3x-10 = (x+\square)(x+\Delta) = (x-2)(x+5)$$

$$\left.\begin{array}{l}\square+\Delta = 3 \\ \square\Delta = -10\end{array}\right\} \begin{array}{l}\square = +5 \\ \Delta = -2\end{array}$$

$$x^2-3x+2 = (x+\square)(x+\Delta) = (x-2)(x+5)$$

$$\left.\begin{array}{l}\square+\Delta = -3 \\ \square\Delta = 2\end{array}\right\} \begin{array}{l}\square = -2 \\ \Delta = -1\end{array}$$

$$f(x) = \frac{(x-2)(x+5)}{(x-2)(x-1)} \Rightarrow \mathcal{D}_f = \mathbb{R} \setminus \{1;2\}$$

Vizsgáljuk a kétoldali határértékeket az $x_1 = 1$ és $x_2 = 2$ pontokban!

$$\lim_{x\to1^+} \frac{(x-2)(x+5)}{(x-2)(x-1)} = \lim_{x\to1^+} \frac{x+5}{x-1} = +\infty \quad \left(\frac{6}{+0}\right)$$

$$\neq \longrightarrow \text{nem megszüntethető szingularitás}$$

$$\lim_{x\to1^-} \frac{(x-2)(x+5)}{(x-2)(x-1)} = \lim_{x\to1^+} \frac{x+5}{x-1} = -\infty \quad \left(\frac{6}{-0}\right)$$

$$\lim_{x\to2^+} \frac{x+5}{x-1} = \frac{7}{1} = 7$$

$$\Big\| \longrightarrow \text{megszüntethető}$$

$$\lim_{x\to2^-} \frac{x+5}{x-1} = \frac{7}{1} = 7$$

4. feladat

$$f(x) = x \sin \frac{1}{x}$$

Legyen $t = 1/x$

$x \to 0^+ \quad t \to +\infty$

$x \to 0^- \quad t \to -\infty$

$$\lim_{t \to \infty} \frac{\sin t}{t} = 0$$

$$\lim_{t \to -\infty} \frac{\sin t}{t} = 0$$

$\Rightarrow \lim_{x \to 0} x \sin \frac{1}{x} = 0$

5. feladat

$$f(x) = \begin{cases} x & \text{ha } |x| < 1 \\ x^2 + ax + b & \text{ha } |x| \geq 1 \end{cases}$$

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} x^2 + ax + b = 1 - a + b$$

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} x = -1 \longrightarrow -1 = 1 - a + b \quad (1)$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = +1 \to 1 = 1 + a + b \quad (2)$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^2 + ax + b = 1 + a + b$$

$(1) + (2) \to 0 = 2 + 2b \to b = -1$

$(2) \to a = -b \to a = 1$

6. feladat

a, $\lim\limits_{x\to 0} \dfrac{\ln \cos 2x}{x^2}$ $\qquad \ln a^b = b\cdot \ln a$

$$\lim_{x\to 0} \frac{\ln\left((\cos^2 2x)^{1/2}\right)}{x^2} = \lim_{x\to 0}\frac{\frac{1}{2}\ln\cos^2 2x}{x^2} =$$

$$= \lim_{x\to 0}\frac{\ln(1-\sin^2 2x)}{2x^2} = \lim_{x\to 0}\frac{\ln(1-\sin^2 2x)}{\sin^2 2x}\cdot\underbrace{\frac{\sin^2 2x}{4x^2}}_{\to 1}\cdot 2$$

$$= \lim_{x\to 0} 2\cdot\ln(1-\sin^2 2x)^{\frac{1}{\sin^2 2x}}$$

$$= 2\cdot\lim_{x\to 0}\ln\underbrace{\left(1-\frac{1}{\underbrace{\frac{1}{1-\sin^2 2x}}_{\to\infty}}\right)^{\overbrace{\frac{1}{\sin^2 2x}}^{\to\infty}}}_{\to e^{-1}} = 2\ln e^{-1} = -2$$

b, $\lim\limits_{x\to \pi/4} \tan^{\tan 2x} x$ $\qquad \tan 2x = \dfrac{2\tan x}{1-\tan^2 x} = \dfrac{-2\tan x}{(\tan x+1)(\tan x-1)}$

$$\tan^{\tan 2x} x = (1+\tan x-1)^{\tan 2x} = \left(1+\frac{1}{\frac{1}{\tan x-1}}\right)^{\tan 2x}$$

$$= \underbrace{\left(1+\frac{1}{\underbrace{\frac{1}{\tan x-1}}_{\to\infty}}\right)^{\overbrace{\frac{1}{\tan x-1}}^{\to\infty}}}_{\to e}\cdot\underbrace{\frac{-2\tan x}{\tan x+1}}_{\to -1}$$

$$\Rightarrow \lim_{x\to \pi/4}\tan^{\tan 2x} x = e^{-1}$$