Matematika G1 - GY5

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1. feladat

$$a_n = \frac{2n+5}{n-1}$$

$$\varepsilon = 10^{-6}$$

$$\lim_{n\to\infty} \frac{2n+5}{n-1} = 2 \Rightarrow \left|\frac{2n+5}{n-1} - 2\right| < \varepsilon$$

$$\left|\frac{2n+5-2n+2}{n-1}\right| < \varepsilon$$

$$\left|\frac{7}{n-1}\right| < \varepsilon$$

$$n(10^{-6}) = \frac{7}{10^{-6}} + 1 = 7.000.001 \Leftarrow \frac{7}{\varepsilon} + 1 < n(\varepsilon)$$

$$b_n = \frac{2n+3\sqrt{n}}{3n+1}$$

$$\varepsilon = 10^{-3}$$

$$\lim_{n\to\infty} b_n = 2/3 \Rightarrow \left|\frac{2n+3\sqrt{n}}{3n+1} - \frac{2}{3}\right| < \varepsilon$$

$$\left|\frac{6n+9\sqrt{n}-6n-2}{9n+3}\right| < \varepsilon$$

$$\left|\frac{9\sqrt{n}-2}{9n+3}\right| < \varepsilon = 0{,}001$$

$$9\sqrt{n} < 0{,}001(9n+3)+2$$

$$81n < (0{,}009n + 2{,}003)^2 = 81\cdot 10^{-6} n + 0{,}036054n + 4{,}012$$

$$n_1 \cong 999554{,}83$$
$$n_2 \cong 0{,}04955$$
$$\Rightarrow n = 999555$$

2. feladat

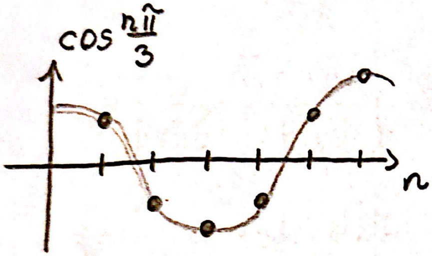
$$a_n = \frac{\left(\sqrt{n^2+1}+n\right)^2}{\sqrt[3]{n^6+1}} \cdot \cos\frac{n\pi}{3}$$

Vizsgáljuk külön a 2 tényezősorozatot!

$$\frac{n^2+1+2n\sqrt{n^2+1}+n^2}{\sqrt[3]{n^6+1}}$$

$n \to \infty$

$$\frac{1+2+1}{1} = 4$$



$\cos \pi/3 = 1/2$

$\cos 2\pi/3 = -1/2$

$\cos 3\pi/3 = -1$

$\cos 4\pi/3 = -1/2$

$\cos 5\pi/3 = 1/2$

$\cos 6\pi/3 = 1$

$\Downarrow$

$\{\pm 1; \pm 1/2\}$

Összegezve: $\{\pm 1\cdot 4; \pm 1/2 \cdot 4\} \to \{-4; -2; 2; 4\}$

$$b_n = \frac{(-1)^n \cdot n + 2n}{3n} \cdot \sin\frac{2n\pi}{3} \to \left\{\pm\frac{\sqrt{3}}{2}; 0\right\}$$

n páros: $(-1)^n = 1 \to \frac{3n}{3n} \to 1$

n páratlan: $(-1)^n = -1 \quad \frac{n}{3n} \to \frac{1}{3}$

$\{1; 1/3\}$

Összegezve

$$\left\{-\frac{\sqrt{3}}{2}; -\frac{\sqrt{3}}{6}; 0; \frac{\sqrt{3}}{6}; \frac{\sqrt{3}}{2}\right\}$$

3. feladat

$$a_n = \frac{2n^2+1}{n^2-n+1}$$

Monotonitás feltétele:

$$a_{n+1} - a_n \overset{?}{\gtrless} 0$$

$$a_{n+1} - a_n = \frac{2(n+1)^2+1}{(n+1)^2-(n+1)+1} - \frac{2n^2+1}{n^2-n+1} = \frac{2n^2+4n+3}{n^2+n+1} - \frac{\dots}{\dots}$$

$$= \frac{(2n^2+4n+3)(n^2-n+1)-(2n^2+1)(n^2+n+1)}{(n^2+n+1)(n^2-n+1)} =$$

$$= \frac{n^4\overset{=0}{(2-2)} + n^3\overset{=0}{(-2+4-2)} + n^2\overset{=-2}{(2+3-4-2-1)} + n\overset{=0}{(4-3-1)} + \overset{=2}{(3-1)}}{n^4(1) + n^3\underset{=0}{(-1+1)} + n^2\underset{=1}{(1+1-1)} + n\underset{=0}{(1-1)} + (1)}$$

$$= \frac{-2n^2+2}{n^4+n^2+1} = \frac{\ominus}{\oplus} < 0 \to$$ szig mon csökken

($n=1$-re 0)

→ határérték: $\lim\limits_{n\to\infty} a_n = 2$ ← alsó korlát

→ első elem: $a_1 = 3$ ← felső korlát

$$b_n = \frac{n^5}{n!}$$

$$\frac{a_{n+1}}{a_n} \overset{?}{\lessgtr} 1$$ → sz.m.cs. / → sz.m.n.

$$\frac{\frac{(n+1)^5}{(n+1)!}}{\frac{n^5}{n!}} = \frac{(n+1)^5}{n^5}\cdot\frac{n!}{(n+1)!} = \left(\frac{n+1}{n}\right)^5 \cdot \frac{n!}{(n+1)n!} = \frac{(n+1)^4}{n^5} < 1$$

$n \geq 4$ esetén

↓

szig mon csökken

$a_0 = 0$ ← alsó korlát

$a_4 = 42,\dot{6}$ ← felső korlát

4. feladat

$a_1 = 1$

$a_n = a_{n-1} + \frac{1}{2^{n-1}}$

$a_1 = 1$

$a_2 = 1 + 1/2$

$a_3 = 1 + 1/2 + 1/4$

$\Downarrow$

Mértani sorozat: $q = 1/2$

$$S_n = a_1 \cdot \frac{1}{1-q} = 1 \cdot \frac{1}{1-1/2} = 2$$

$q < 1$ esetén!!!

5. feladat

$a_1 = 1$

$a_n = \sqrt{1 + a_{n-1}}$

Monotonitás:

$a_1 = 1 \quad a_2 = \sqrt{2} \quad a_3 \simeq 1{,}554$

$\Rightarrow$ monoton nő

Korlátosság: Kell egy K szám, hogy ha $a_n \leq K$, akkor $a_{n+1} \leq K$

$$a_{n+1} = \sqrt{1 + a_n} \leq \sqrt{1+K} \leq K$$

$$1 + K \leq K^2$$

$$0 \leq K^2 - K - 1$$

$$K_{1,2} = \frac{1 \pm \sqrt{5}}{2} \simeq \begin{cases} 1{,}618 \\ -0{,}618 \end{cases}$$

Felső korlát

Határérték: $\sqrt{1+A} = A$

$$1 + A = A^2 \rightarrow A = \frac{1+\sqrt{5}}{2}$$

6. feladat

$$a_n = \frac{n^2 - i(n^2-1)}{n^2 - i} \overset{\substack{\text{bővítés} \\ \left(\frac{n^2+i}{n^2+i}\right)\text{-vel}}}{=} \frac{n^4 - i(n^4-n^2) + in^2 + (n^2-1)}{n^4+1} =$$

$$= \frac{n^4+n^2-1}{n^4+1} + i\,\frac{-n^4+2n^2}{n^4+1} \xrightarrow[n\to\infty]{} 1-i$$

$$b_n = \frac{i^n}{3^n + i^n} = \frac{(i/3)^n}{1^n + (i/3)^n} \xrightarrow[n\to\infty]{} \frac{0}{1+0} = 0$$

$c_n = (1-i)^n$ → divergens, mert $|1-i| = \sqrt{2} > 1$

7. feladat

$n^{n+1} \geq (n+1)^n$, ha $n \geq 3$

$n \cdot n^n \geq (n+1)^n$

$$n \geq \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n \xrightarrow[n\to\infty]{} e$$

$n=1 < e$

$n=2 < e$

$n=3 > e$ ✓

8. feladat

$$a_n = \binom{n}{2} \Big/ \binom{n}{3} = \frac{\frac{n!}{(n-2)!\,2!}}{\frac{n!}{(n-3)!\,3!}} = \frac{(n-3)!\,3!}{(n-2)!\,2!} = \frac{6}{2(n-2)}$$

$n > 3$!!

$n \not\to \infty$

$\swarrow$

0

Monotonitás: $a_{n+1} - a_n \gtrless 0$

$$\frac{6}{2(n+1-2)} - \frac{6}{2(n-2)} = 3\left(\frac{1}{n-1} - \frac{1}{n-2}\right) = 3 \cdot \frac{(n-2)-(n-1)}{(n-1)(n-2)} =$$

$$= 3 \cdot \frac{-1}{n^2-3n+2} = \frac{\ominus}{\oplus} \rightarrow \text{szig mon cs.}$$

Korlátosság: $K = 3 \quad k = 0$