Matematika G1 - GY4 Sándor Tibor

1. feladat

a, $a_n = \left|\frac{n+1}{3n-8}\right|$ def: $|a_n - a| < \varepsilon$, ha $n > N(\varepsilon)$

$\left|\frac{n+1}{3n-8} - A\right| < \varepsilon$ Sejtsük meg A-t!

$\lim_{n\to\infty} \frac{n/n + 1/n}{3n/n - 8/n} = \frac{1}{3} \to A = 1/3$

Visszahelyettesítve:

$\left|\frac{n+1}{3n-8} - \frac{1}{3}\right| = \left|\frac{3n+3-3n+8}{9n-24}\right| = \left|\frac{11}{9n-24}\right| < \varepsilon$

$\Leftrightarrow \frac{11}{9\varepsilon} + \frac{11}{24} < n$

b, $\frac{n\cdot(-1)^n - 1}{2n} \overset{def}{\Longrightarrow} \left|\frac{(-1)^n}{2} - \frac{1}{2n} - A\right| < \varepsilon$

- n páros: $\left|\frac{1}{2} + \frac{1}{2n} - A\right| < \varepsilon$
- n páratlan: $\left|\frac{-1}{2} + \frac{1}{2n} - A\right| < \varepsilon$

$\Rightarrow$ nem konvergens

2. feladat

$a_n = \frac{n^2 - 6n + 7}{n^2 + 12n + 49}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2/n^2 - 6n/n^2 + 7/n^2}{n^2/n^2 + 12n/n^2 + 49/n^2} = \frac{1+0+0}{1+0+0} = \frac{1}{1} = 1$

$$b_n = \frac{1+2+\dots+n}{n+2} - \frac{n}{2} = \frac{n(n+1)/2}{n+2} - \frac{n}{2} = \frac{n(n+1)-n(n+2)}{2(n+2)}$$

$$= \frac{n^2+n-n^2-2n}{2n+4} = \frac{-n}{2n+4} \xrightarrow{n \to \infty} \frac{-1}{2}$$

$$c_n = \frac{(-2)^n+3^n}{(-2)^{n+1}+3^{n+1}} = \frac{(-1)^n(2/3)^n + 3^n/3^n}{-2\cdot(2/3)^n+3\cdot 3^n/3^n} \xrightarrow{n \to \infty} \frac{1}{3}$$

$$d_n = \frac{\sqrt{n^3+3n^2}+\sqrt[3]{n^4+1}}{\sqrt[4]{5n^6+2}+\sqrt[5]{n^7+3n^3}} + \frac{n!}{(n+1)!+3^{2n}} \xrightarrow{n \to \infty} \frac{1}{\sqrt[4]{5}}$$

domináns: $n^{3/2} = n^{6/4}$

$1/\sqrt[4]{5}$

nevező gyorsabban nő

0

$$e_n = \underbrace{\sqrt{n^2+n+1}}_{a} - \underbrace{\sqrt{n^2-n+1}}_{b} \cdot \left(\frac{a+b}{a+b}\right) = \frac{\overbrace{n^2+n+1}^{a^2} - \overbrace{(n^2-n+1)}^{b^2}}{\sqrt{n^2+n+1}+\sqrt{n^2-n+1}}$$

$$= \frac{2n}{\sqrt{n^2+n+1}+\sqrt{n^2-n-1}} \xrightarrow{n \to \infty} \frac{2}{1+1} = 1$$

domináns: $n = \sqrt{n^2}$

$$f_n = \sqrt[3]{n^2-n^3} + n = \frac{n^2-n^3+n^3}{(n^2-n^3)^{2/3} - n\sqrt[3]{n^2-n^3} + n^2} =$$

$(a \pm b)(a^2 \mp ab + b^2) = a^3 \pm b^3$

$a = \sqrt[3]{n^2-n^3}$

$b = n$

$$= \frac{n^2}{(n^4-2n^5+n^6)^{\frac{1}{3}} - (n^5-n^6)^{\frac{1}{3}} + n^2}$$

domináns: $n^2 = n^{6/3}$

$$\xrightarrow{n \to \infty} \frac{1}{1+1+1} = \frac{1}{3}$$

3. feladat

$$\frac{1}{3^n} < \frac{n}{3^n} < \frac{2^n}{3^n} \quad \checkmark$$

$$\downarrow \quad 0$$

4. feladat

a) $$\lim_{n\to\infty} \sqrt[n]{5n^2-30n-21} = \lim_{n\to\infty} e^{\ln \sqrt[n]{5n^2-30n-21}} =$$

$$= \lim_{n\to\infty} e^{\frac{\ln(5n^2-30n-21)}{n}} = e^0 = 1$$

n gyorsabban nő mint $\ln(\dots)$

b) $$\lim_{n\to\infty} \underbrace{\left[\frac{\cos n^3}{2n} - \frac{3n}{6n+1}\right]}_{} \qquad -1 \leq \cos(\dots) \leq 1$$

$$\frac{-1}{2n} - \frac{3n}{6n+1} \leq a_n \leq \frac{+1}{2n} - \frac{3n}{6n+1}$$

$$\frac{-6n-1-6n^2}{12n^2+2n} \leq a_n \leq \frac{6n+1-6n^2}{12n^2+2n}$$

$$-1/2 \leq a_n \leq -1/2 \Rightarrow a_n \to -1/2$$

c) $$\lim_{n\to\infty} \left(1+\frac{1}{n^2}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n^2}\right)^{n^2 \cdot \frac{1}{n}} = \lim_{n\to\infty} e^{1/n} = e^0 = 1$$

d) $$\lim_{n\to\infty} \left(\frac{3n-1}{3n+2}\right)^{2n} = \lim_{n\to\infty} \left(\frac{3n+2-3}{3n+2}\right)^{2n} = \lim_{n\to\infty} \left(1-\frac{3}{3n+2}\right)^{2n}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{\frac{3n+2}{-3}}\right)^{\frac{3n+2}{-3} \cdot \frac{-3}{3n+2} \cdot 2n} = e^{-2}$$

e, $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{\ln n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n\cdot\frac{\ln n}{n}} = e^0 = 1$

f, $\lim_{n\to\infty}\left(\frac{n^2-n+1}{n^2+n+1}\right)^{2n+5} = \lim_{n\to\infty}\left(\frac{n^2+n+1-2n}{n^2+n+1}\right)^{2n+5}$

$= \lim_{n\to\infty}\left(1+\frac{-2n}{n^2+n+1}\right)^{2n+1} = \lim_{n\to\infty}\left(1+\frac{1}{\frac{n^2+n+1}{-2n}}\right)^{2n+1}$

$= \lim_{n\to\infty}\left(1+\frac{1}{\frac{n^2+n+1}{-2n}}\right)^{\frac{n^2+n+1}{-2n}\cdot\frac{(2n+1)(-2n)}{n^2+n+1}} = e^{-4}$

5. feladat

$\lim_{n\to\infty}\left(1+\frac{k}{n}\right)^n = e^k$

-> Teljes indukció:

$k=0 \to (1+0)^n \to e^0$

$k=1 \to (1+1/n)^n \to e$ (def)

-> TFH $\forall k$-ra igaz!

-> Vizsgáljuk $(k+1)$-re

$\lim_{n\to\infty}\left(1+\frac{k+1}{n}\right)^n = \lim_{n\to\infty}\left(\frac{n+k+1}{n}\right)^n =$

$= \lim_{n\to\infty}\left(\frac{n+1+k}{n+1}\right)^{n+1}\cdot\frac{(n+1)^n}{n^n}\cdot\frac{n+1}{n+1+k} =$

$= \lim_{n\to\infty}\left(1+\frac{k}{n+1}\right)^{n+1}\cdot\left(1+\frac{1}{n}\right)^n\cdot\frac{n+1}{n+1+k} = e^k\cdot e\cdot 1 = e^{k+1}$ ✓