Matematika G1- GY1 — Sándor Tibor

1. feladat

a) $\begin{cases} \underline{u} = 2\underline{a} + 3\underline{b} \\ \underline{v} = 4\underline{a} + \alpha\underline{b} \end{cases}$

kollinearisak: $\underline{u} = \lambda\underline{v}$

$2\underline{a} + 3\underline{b} = \lambda(4\underline{a} + \alpha\underline{b})$

$\{\underline{a}; \underline{b}\}$ lin független $\rightarrow$ $\begin{cases} (1)\ 2\underline{a} = \lambda 4\underline{a} \\ (2)\ 3\underline{b} = \lambda\alpha\underline{b} \end{cases}$

(1) $\rightarrow 2 = \lambda \cdot 4 \rightarrow \lambda = 1/2$

(2) $\rightarrow 3 = \lambda\alpha \rightarrow 3 = 1/2 \cdot \alpha \rightarrow \underline{\alpha = 6}$

b) $\begin{cases} \underline{u} = 3\underline{a} - 3\alpha\underline{b} + \beta\underline{c} \\ \underline{v} = \underline{a} - \alpha\underline{b} - \underline{c} \end{cases}$

$\underline{u} = \lambda\underline{v}$

$3\underline{a} - 3\alpha\underline{b} + \beta\underline{c} = \lambda(\underline{a} - \alpha\underline{b} - \underline{c})$

$\begin{cases} (1)\ 3\underline{a} = \lambda\underline{a} \\ (2)\ -3\alpha\underline{b} = \lambda(-\alpha\underline{b}) \\ (3)\ \beta\underline{c} = \lambda(-\underline{c}) \end{cases}$

(1) $\rightarrow \lambda = 3$

(2) $\rightarrow -3\alpha = -3\alpha \rightarrow \underline{\alpha \in \mathbb{R}}$ (tetszőleges)

(3) $\rightarrow \underline{\beta = -3}$

2. feladat

a) $\begin{cases} \underline{r} = 3\underline{a} + 2\underline{b} + \underline{c} \\ \underline{s} = 5\underline{a} - 3\underline{b} - 2\underline{c} \\ \underline{t} = \underline{0} \end{cases}$ $\rightarrow \underline{t} = \underline{0} \rightarrow$ lineárisan összefüggőek

b) $\begin{cases} \underline{r} = \underline{a} + \underline{b} + \underline{c} \\ \underline{s} = \underline{b} + \underline{c} \\ \underline{t} = \underline{a} + \underline{c} \end{cases}$

$\lambda_1\underline{r} + \lambda_2\underline{s} + \lambda_3\underline{t} = \underline{0}$

$\lambda_1(\underline{a} + \underline{b} + \underline{c}) + \lambda_2(\underline{b} + \underline{c}) + \lambda_3(\underline{a} + \underline{c}) = \underline{0}$

$\begin{cases} (a)\ \lambda_1 + \lambda_3 = 0 \\ (b)\ \lambda_1 + \lambda_2 = 0 \\ (c)\ \lambda_1 + \lambda_2 + \lambda_3 = 0 \end{cases}$

$\left.\begin{array}{l} (c) - (a) \rightarrow \lambda_2 = 0 \\ (b) \rightarrow \lambda_1 = 0 \\ (a) \rightarrow \lambda_3 = 0 \end{array}\right\}$ $\lambda_1 = \lambda_2 = \lambda_3 = 0$

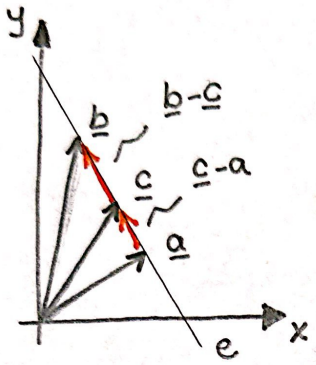
$\Downarrow$

lineárisan függetlenek

3. feladat

$\{\underline{a}, \underline{b}, \underline{c}\}$ komplanáris → egy síkon vannak

$\underline{a}$ és $\underline{b}$ nem kollineáris → $\nexists \lambda \in \mathbb{R}$, hogy $\underline{a} = \lambda \underline{b}$



Szükségesség:

TFH egy egyenesen vannak:

ekkor $\exists \lambda \in \mathbb{R}$, hogy $\lambda(\underline{c} - \underline{a}) = \underline{b} - \underline{c}$

$$(\lambda+1)\underline{c} = \lambda \underline{a} + \underline{b}$$

$$\underline{c} = \underbrace{\frac{\lambda}{\lambda+1}}_{\alpha}\underline{a} + \underbrace{\frac{1}{\lambda+1}}_{\beta}\underline{b} \qquad (\lambda \neq -1)$$

$$\alpha + \beta = \frac{\lambda}{\lambda+1} + \frac{1}{\lambda+1} = \frac{\lambda+1}{\lambda+1} = 1 \quad \checkmark$$

Elégségesség:

TFH $\alpha + \beta = 1$, ekkor $\underline{c} = \alpha \underline{a} + (1-\alpha)\underline{b}$.

$$\underline{c} - \underline{a} = (\alpha - 1)\underline{a} + (1-\alpha)\underline{b}$$

$$\underline{b} - \underline{c} = -\alpha \underline{a} + \alpha \underline{b} = \alpha(\underline{b} - \underline{a})$$

↙ $\alpha = 0$ ↘ $\alpha \neq 0$

$\underline{b} - \underline{c} = \underline{0}$ ✓

$$\underline{b} = \frac{\underline{c} - \alpha \underline{a}}{1-\alpha}$$

$$\underline{b} - \underline{c} = \alpha\left(\frac{\underline{c}}{1-\alpha} + \frac{-\alpha}{1-\alpha}\underline{a} + \frac{\alpha-1}{1-\alpha}\underline{a}\right)$$

$$\underline{b} - \underline{c} = \frac{\alpha}{1-\alpha}(\underline{c} - \underline{a})$$

→ $\exists \lambda \in \mathbb{R}$ ✓

4. feladat

$$\underline{a} = \begin{bmatrix} 7 \\ -1 \\ 6 \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} 2 \\ 20 \\ 2 \end{bmatrix}$$

$$|\underline{a}| = \sqrt{7^2 + (-1)^2 + 6^2} = \sqrt{86}$$

$$|\underline{b}| = \sqrt{2^2 + 20^2 + 2^2} = \sqrt{408}$$

$$\underline{a} \cdot \underline{b} = |\underline{a}| \cdot |\underline{b}| \cdot \cos\varphi$$

$$\cos\varphi = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}| \cdot |\underline{b}|} = \frac{6}{\sqrt{86}\sqrt{408}}$$

$$\varphi \simeq 88{,}16°$$

$$\underline{a} \cdot \underline{b} = 7 \cdot 2 + (-1)20 + 6 \cdot 2 = 6$$

5. feladat

$\underline{a} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ $\underline{b} = \begin{bmatrix} 6 \\ -2 \\ z \end{bmatrix}$

$\underline{a} \perp \underline{b} \rightarrow \underline{a} \cdot \underline{b} = 0$

$2 \cdot 6 + (-3)(-2) + z = 0$

$\underline{z = -18}$

6. feladat

$\underline{a} + 3\underline{b} \perp 7\underline{a} - 5\underline{b}$

$(\underline{a} + 3\underline{b}) \cdot (7\underline{a} - 5\underline{b}) = 0$

$7\underline{a}^2 + 21\underline{a} \cdot \underline{b} - 5\underline{a} \cdot \underline{b} - 15\underline{b}^2 = 0$

$16\underline{a} \cdot \underline{b} = -7\underline{a}^2 + 15\underline{b}^2$

$\underline{a} - 4\underline{b} \perp 7\underline{a} - 2\underline{b}$

$(\underline{a} - 4\underline{b}) \cdot (7\underline{a} - 2\underline{b}) = 0$

$7\underline{a}^2 - 28\underline{a} \cdot \underline{b} - 2\underline{a} \cdot \underline{b} + 8\underline{b}^2 = 0$

$30\underline{a} \cdot \underline{b} = 7\underline{a}^2 + 8\underline{b}^2$

$\frac{1}{16}(-7\underline{a}^2 + 15\underline{b}^2) = \frac{1}{30}(7\underline{a}^2 + 8\underline{b}^2)$

$-210\underline{a}^2 + 450\underline{b}^2 = 112\underline{a}^2 + 128\underline{b}^2$

$332\underline{b}^2 = 332\underline{a}^2$

$\underline{a}^2 = \underline{b}^2$ (1.) $\rightarrow \underline{a} = \underline{b} = 0$ ↯

$\underline{a} \cdot \underline{b} = \frac{1}{16}(-7 + 15)\underline{a}^2 = \frac{\underline{a}^2}{2}$

$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos\varphi \rightarrow \cos\varphi = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}| \cdot |\underline{b}|} = \frac{\underline{a}^2/2}{|\underline{a}||\underline{a}|} = \frac{1}{2}$

7. feladat

$A(2;6;0)$

$B(1;2;3)$

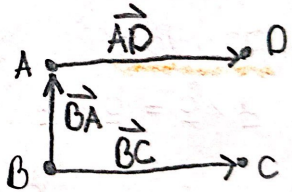
$C(-2;8;z)$

$\vec{BA} = \underline{a} - \underline{b} = (1;4;-3)$

$\vec{BC} = \underline{c} - \underline{b} = (-3;6;z-3)$

$\vec{BA} \perp \vec{BC} \rightarrow \vec{BA} \cdot \vec{BC} = 0 = -3 + 24 - 3z + 9$

$z = 10$



$\vec{BC} = (-3;6;7)$

$\vec{AD} = \vec{BC} \rightarrow D = A + \vec{AD} = (2;6;0) + (-3;6;7)$

$D = (-1;12;7)$

8. feladat

$$\underline{a}=\begin{bmatrix}-4\\2\\1\end{bmatrix} \quad \underline{b}=\begin{bmatrix}-2\\7\\8\end{bmatrix}$$

$$\underline{a}\times\underline{b}=\begin{bmatrix}-4\\2\\1\end{bmatrix}\times\begin{bmatrix}-2\\7\\8\end{bmatrix}=\begin{bmatrix}2\cdot 8-1\cdot 7\\1(-2)-(-4)8\\-4\cdot 7-2(-2)\end{bmatrix}=\begin{bmatrix}9\\30\\-24\end{bmatrix}$$

9. feladat

$$(3\underline{a}-\underline{b})\times(\underline{b}+3\underline{a})= 3(\underline{a}\times\underline{b})-\underbrace{\underline{b}\times\underline{b}}_{=\underline{0}}+9\underbrace{(\underline{a}\times\underline{a})}_{=\underline{0}}-3(\underline{b}\times\underline{a})$$

$$=3(\underline{a}\times\underline{b})+3(\underline{a}\times\underline{b})=6(\underline{a}\times\underline{b})$$

10. feladat

$$\underline{a}=\begin{bmatrix}-3\\4\\7\end{bmatrix} \quad \underline{b}=\begin{bmatrix}2\\5\\1\end{bmatrix} \quad \underline{a}\times\underline{b}=\begin{bmatrix}-3\\4\\7\end{bmatrix}\times\begin{bmatrix}2\\5\\1\end{bmatrix}=\begin{bmatrix}4\cdot 1-7\cdot 5\\7\cdot 2-(-3)1\\-3\cdot 5-4\cdot 2\end{bmatrix}=\begin{bmatrix}-31\\17\\-23\end{bmatrix}$$

11. feladat

A(1;0;2)
B(-1;4;7)
C(5;-2;1)

$$\vec{AC}=C-A=(4;-2;-1)$$
$$\vec{AB}=B-A=(-2;4;5)$$

$$|\vec{AB}\times\vec{AC}|=2T_{\triangle}$$

$$\vec{AB}\times\vec{AC}=\begin{bmatrix}4\\-2\\-1\end{bmatrix}\times\begin{bmatrix}-2\\4\\5\end{bmatrix}=\begin{bmatrix}-2\cdot 5-(-1)4\\-1(-2)-4\cdot 5\\4\cdot 4-(-2)(-2)\end{bmatrix}=\begin{bmatrix}-6\\-18\\12\end{bmatrix}$$

$$2T=\sqrt{6^2+18^2+12^2}=6\sqrt{14}$$

$$T=3\sqrt{14}$$

12. feladat

$$\underline{a}\times\underline{c}=\underline{b}\times\underline{c} \qquad \underline{a}\overset{?}{=}\underline{b}$$

$$\underline{a}\times\underline{c}-\underline{b}\times\underline{c}=\underline{0} \rightarrow (\underline{a}-\underline{b})\times\underline{c}=\underline{0} \longrightarrow \underline{a}-\underline{b}=\underline{0}$$

$$\hookrightarrow \underline{c}=\underline{0}$$

$$\hookrightarrow \underline{a}-\underline{b}\parallel\underline{c}$$

13. feladat

$\underline{a} = \begin{bmatrix} 6 \\ 2 \\ -3 \end{bmatrix}$ $\underline{b} = \begin{bmatrix} -3 \\ 6 \\ -2 \end{bmatrix}$

$\underline{a} \overset{?}{\perp} \underline{b}$ $\quad \underline{a} \cdot \underline{b} \overset{?}{=} 0$

$6(-3) + 2 \cdot 6 + (-3)(-2) = 0$ ✓

$\pm 2\underline{c} = \underline{a} \times \underline{b} = \begin{bmatrix} 6 \\ 2 \\ -3 \end{bmatrix} \times \begin{bmatrix} -3 \\ 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 14 \\ 21 \\ 42 \end{bmatrix} = 7 \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$ $\quad \underline{c} = \pm \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$

14. feladat

$\underline{a}(2;3;-1)$
$\underline{b}(1;-1;3)$
$\underline{c}(1;9;-11)$

$\underline{b} \times \underline{c} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \times \begin{bmatrix} 1 \\ 9 \\ -11 \end{bmatrix} = \begin{bmatrix} -16 \\ 14 \\ 10 \end{bmatrix}$

$\underline{a} \cdot (\underline{b} \times \underline{c}) = 2(-16) + 3(14) + (-1)10 = 0$

⇓

lineárisan összefüggenek

15. feladat

$\underline{r} = 2\underline{a} + 3\underline{b} + 4\underline{c}$
$\underline{s} = \underline{a} - \underline{b} + \underline{c}$
$\underline{t} = 2\underline{a} + 4\underline{b} - \underline{c}$

$\underline{r} \cdot (\underline{s} \times \underline{t}) = (2\underline{a} + 3\underline{b} + 4\underline{c}) \cdot [(\underline{a} - \underline{b} + \underline{c}) \times (2\underline{a} + 4\underline{b} - \underline{c})]$

$= (2\underline{a} + 3\underline{b} + 4\underline{c}) \cdot [2\underline{a} \times \underline{a} + 4\underline{a} \times \underline{b} - \underline{a} \times \underline{c}$
$- 2\underline{b} \times \underline{a} - 4\underline{b} \times \underline{b} + \underline{b} \times \underline{c}$
$+ 2\underline{c} \times \underline{a} + 4\underline{c} \times \underline{b} - \underline{c} \times \underline{c}]$

$= (2\underline{a} + 3\underline{b} + 4\underline{c}) \cdot [+6\underline{a} \times \underline{b} - 3\underline{a} \times \underline{c} - 3\underline{b} \times \underline{c}]$

$= 24\,\underline{cab} - 9\underline{bac} - 6\underline{abc} = (24 + 9 - 6)\,\underline{abc} = \underline{27V}$

$\underline{cab} \rightarrow 24\,\underline{abc}$ $\quad \underline{bac} \rightarrow +9\underline{abc}$

16. feladat

$\underline{a}\,(3;\alpha;0)$
$\underline{b}\,(0;3;\alpha)$
$\underline{c}\,(1;0;-1)$

$\underline{abc} = 0$

$$\underline{b}\times\underline{c} = \begin{bmatrix}0\\3\\\alpha\end{bmatrix} \times \begin{bmatrix}1\\0\\-1\end{bmatrix} = \begin{bmatrix}-3\\\alpha\\-3\end{bmatrix}$$

$$\underline{a}\cdot(\underline{b}\times\underline{c}) = 3(-3)+\alpha^2+0(-3)$$

$$\alpha^2 = 9$$

$\underline{\alpha = \pm 3}$ → ekkor lesz 2in összefüggő

17. feladat

$\underline{a}\,(-1;2;1)$
$\underline{b}\,(1;2;2)$

$\hat{e}_b = \underline{b}/b \quad b = \sqrt{1^2+2^2+2^2} = 3$

$\hat{e}_b = (1/3 \;\; 2/3 \;\; 2/3)$

$$\underline{a}_{\parallel} = (\underline{a}\cdot\hat{e}_b)\hat{e}_b = \left(-1\cdot\frac{1}{3}+2\cdot\frac{2}{3}+1\cdot\frac{2}{3}\right)\begin{bmatrix}1/3\\2/3\\2/3\end{bmatrix}$$

$$= \frac{5}{3}\begin{bmatrix}1/3\\2/3\\2/3\end{bmatrix} = \begin{bmatrix}5/9\\10/9\\10/9\end{bmatrix}$$

$$\underline{a}_{\perp} = \underline{a}-\underline{a}_{\parallel} = \begin{bmatrix}-1-5/9\\2-10/9\\1-10/9\end{bmatrix} = \begin{bmatrix}-14/9\\8/9\\-1/9\end{bmatrix}$$